

How special is your Aronszajn tree?

Chaz Schlindwein
Division of Mathematics and Computer Science
Lander University
Greenwood, South Carolina 29649, USA
`chaz@lander.edu`

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Abstract

We answer a question of Shelah by constructing a model of Suslin's Hypothesis in which there is an Aronszajn tree T such that for every unbounded $E \subseteq \omega_1$ we have that T is not E -special. We may require that CH holds, or that CH fails, or that Kurepa's hypothesis holds or fails, or that there is a stationary $S \subseteq \omega_1$ such that every Aronszajn tree is S -*-special, or other variants.

1 Introduction

In this note, we answer the following questions of Shelah [Sh, Remark IX.4.9(5)]:

Q1. *Is it consistent that Suslin's Hypothesis holds yet there is an Aronszajn tree T such that for every unbounded $E \subseteq \omega_1$ we have that T is not E -special?*

Q2. *Is it consistent that there is a stationary $S \subseteq \omega_1$ such that every Aronszajn tree is S -*-special yet there is an Aronszajn tree T such that for every unbounded $E \subseteq \omega_1$ we have that T is not E -special?*

Notice that a positive answer to Q2 yields a positive answer to Q1 (see [PIF, Claim IX.3.4(1)] or Lemma 2.9, below).

The intent of Shelah's questions is to delineate the distinction between two different notions of “special” for Aronszajn trees. Shelah [Sh, Remark IX.4.9(2)] addresses this distinction as follows:

Theorem (Shelah). *If ZFC is consistent, then so is ZFC plus there is an unbounded $E \subseteq \omega_1$ such that every Aronszajn tree is E -special and there is an Aronszajn tree T such that for every stationary $S \subseteq \omega_1$ we have that T is not S -*-special.*

This shows in a strong way that E -specialness does not entail S -*-specialness. Thus it is natural to consider whether S -*-specialness entails E -specialness. This is the motivation for the two questions of Shelah given above. We show that the answer to each of the above questions is positive; that is, neither version of “specialness” implies the other. In particular, we demonstrate the following theorem.

Theorem. *If ZFC is consistent, then so is ZFC plus there is a stationary $S \subseteq \omega_1$ such that every Aronszajn tree is S -*-special and there is an Aronszajn tree T such that for every unbounded $E \subseteq \omega_1$ we have that T is not E -special.*

We produce models with certain additional properties. For example, we give a model exemplifying a positive answer to Q1 in which T has no stationary antichain. The models we use are variations of models that have appeared in [S], [S], [S]. We show that each of these models satisfies the statement: “there is no unbounded $E \subseteq \omega_1$ such that T is E -special (in some of the cited papers, the distinguished Aronszajn tree is denoted T^* rather than T).” The innovation which

leads to our answering Shelah's questions is our formulation of a new preservation property (see Definition 3.6). Any forcing that satisfies this property maintains the non- E -specialness (for every unbounded $E \subseteq \omega_1$) of some appropriate Suslin tree of the ground model.

Notations. We say that a map f is order-preserving iff $x \leq y$ implies $f(x) \leq f(y)$, whereas we say that f is strictly order-preserving iff $x < y$ implies $f(x) < f(y)$. For T a tree we let $T_\beta = \{x \in T : \text{rk}(x) = \beta\}$ and for E a set of ordinals we let $T_E = \bigcup \{T_\beta : \beta \in E\}$ (the fact that each ordinal is literally a set of ordinals renders this ambiguous, but it is always clear in context). We say that T is an ω_1 -tree iff every level of T is countable and for every $x \in T$, whenever $\text{rk}(x) < \beta < \omega_1$ then there are at least two successors of x in T_β , and each node whose rank is a limit ordinal is uniquely determined by its set of predecessors.

2 Specializations of ω_1 -trees

There are various ways in which an Aronszajn tree T can be specialized. The classical notion is that T is special iff there is a strictly order-preserving function from T into \mathbf{Q} . The essential point is that a special Aronszajn tree cannot be a Suslin tree. Baumgartner [B] and Shelah [S, chapter IX] investigate weaker notions of “special” that also ensure non-Suslinity. Of particular interest are the following two definitions.

Definition 2.1. Suppose T is an Aronszajn tree and $E \subseteq \omega_1$ is unbounded. We say that T is E -special iff there is a strictly order-preserving map from T_E into \mathbf{Q} .

Definition 2.2. Suppose T is an Aronszajn tree and S is a subset of ω_1 consisting of limit ordinals. We say that T is S -*-special iff there is a function f mapping T_S into ω_1 such that $(\forall x \in T_S)(f(x) < \text{rk}(x))$ and whenever $x < y$ are in T_S then $f(x) \neq f(y)$.

Lemma 2.3. Suppose T is an Aronszajn tree and either T is E -special for some unbounded E or T is S -*-special for some stationary S consisting of limit ordinals. Then T is not Suslin.

Proof: In the first case, let f from T_E into \mathbf{Q} be a specializing function. For some $r \in \mathbf{Q}$ we have that $f^{-1}(r)$ is uncountable; necessarily $f^{-1}(r)$ is an antichain

of T . In the second case, suppose f is as in Definition 2.2. For each $\alpha \in S$ let $g(\alpha) = \min\{f(x) : x \in T_\alpha\}$. By Fodor's theorem we may choose γ such that $g^{-1}(\gamma)$ is uncountable. Clearly $\{x \in T_S : f(x) = \gamma\}$ is an uncountable antichain of T .

In the next two Definitiona the two notions of “special” introduced above are extended to ω_1 -trees (see also [Sch]). We show in Lemma 2.9 that an ω_1 -tree that is special in either of these two extended senses is neither Suslin nor Kurepa. We also show in Lemma 2.8 that for Aronszajn trees, the two extended definitions coincide with the earlier definitions. Until then, we shall specify, e.g., “ E -special in the sense of Definition 2.4.”

Definition 2.4. Suppose T is an ω_1 -tree and $E \subseteq \omega_1$ is uncountable. We say that T is E -special iff there is an order-preserving f mapping T_E into \mathbf{Q} such that whenever $\{x, y, z\} \subseteq T_E$ and $f(x) = f(y) = f(z)$ and $x < y$ and $x < z$, then y and z are comparable.

Definition 2.5. Suppose T is an ω_1 -tree and S is a subset of ω_1 consisting of limit ordinals. We say that T is S -*-special iff there is a function f mapping T_S into ω_1 such that $(\forall x \in T_S)(f(x) < \text{rk}(x))$ and whenever $\{x, y, z\} \subseteq T_S$ and $f(x) = f(y) = f(z)$ and $x < y$ and $x < z$ then y and z are comparable.

Lemma 2.6. Suppose T is an ω_1 -tree and S_1 and S_2 are subsets of ω_1 consisting of limit ordinals and the symmetric difference $S_1 \Delta S_2 = (S_1 - S_2) \cup (S_2 - S_1)$ is nonstationary. Then T is S_1 -*-special in the sense of Definition 2.5 iff T is S_2 -*-special in the sense of Definition 2.5.

Proof: It suffices to show that whenever C is a closed unbounded set and T is $(S \cap C)$ -*-special in the sense of Definition 2.5, then T is S -*-special in the sense of Definition 2.5. Let f mapping $T_{S \cap C}$ into ω_1 be a specializing function. For $x \in T_{S \cap C}$, let γ_x be a limit ordinal (or zero) and n_x an integer such that $f(x) = \gamma_x + n_x$. Because $S - C$ is a non-stationary set of limit ordinals, we may take h to be a one-to-one function from $S - C$ into ω_1 such that $(\forall \alpha \in S - C)(h(\alpha) < \alpha)$. We may assume that the range of h consists only of odd ordinals. Define g such that for $x \in T_{S \cap C}$ we have that $g(x) = \gamma_x + 2n_x$, and for $x \in T_{S - C}$ we have $g(x) = h(\text{rk}(x))$. Clearly g demonstrates that T is S -*-special in the sense of Definition 2.5. The Lemma is established.

Lemma 2.7. *Suppose T is an Aronszajn tree and $E \subseteq \omega_1$ is uncountable. Then T is E -special in the sense of Definition 2.1 iff there is g mapping T_E into ω such that whenever $x < y$ are in T_E then $g(x) \neq g(y)$. Furthermore, T is E -special in the sense of Definition 2.4 iff there is h mapping T_E into ω such that whenever $x < y$ are in T_E and $z \in T_E$ and $x < z$ and $g(x) = g(y) = g(z)$ then y is comparable with z .*

Proof: The “only if” direction of the first assertion is evident by considering the composition of a specializing function with a one-to-one mapping of \mathbf{Q} into ω . For the “if” direction, suppose g is given. Build $\langle f_n : n \in \omega \rangle$ by recursion such that $\text{dom}(f_n) = \{x \in T_E : g(x) \leq n\}$ and $\text{range}(f_n)$ is a finite subset of \mathbf{Q} and whenever $x < y$ are in $\text{dom}(f_n)$ then $f_n(x) < f_n(y)$. There is no difficulty in doing this. Clearly $\bigcup \{f_n : n \in \omega\}$ is an E -specializing function in the sense of Definition 2.1. The first assertion is established.

The “only if” direction of the second assertion is again easy to see by considering the composition of an E -specializing function with a mapping from \mathbf{Q} into ω . For the “if” direction, given g as in the statement of the assertion, then for every $m \in \omega$ let \mathcal{I}_m be the set of minimal elements of $g^{-1}(m)$, and for every $x \in \mathcal{I}_m$ use the fact that T is Aronszajn to choose $\langle y_{m,x,i} : i \in \omega \rangle$ an enumeration of $\{y \in T_E : x \leq y \text{ and } g(y) = m\}$. Build $\langle f_n : n \in \omega \rangle$ by recursion such that $\text{dom}(f_n) = \{y \in T_E : (\exists m \leq n)(\exists x \in \mathcal{I}_m)(\exists i \leq n)(y = y_{m,x,i})\}$, and $\text{range}(f_n)$ is a finite subset of \mathbf{Q} and whenever $u < v$ and $u < w$ are all in $\text{dom}(f_n)$ and $f_n(u) = f_n(v) = f_n(w)$ then v is comparable with w . There is again no difficulty in doing this. Clearly $\bigcup \{f_n : n \in \omega\}$ is an E -specializing function in the sense of Definition 2.2. The Lemma is established.

Lemma 2.8. *Suppose T is Aronszajn and $E \subseteq \omega_1$ is unbounded. Then T is E -special in the sense of Definition 2.1 iff T is E -special in the sense of Definition 2.4, and for $S \subseteq \omega_1$ consisting of limit ordinals we have that T is S -*-special in the sense of Definition 2.2 iff T is S -*-special in the sense of Definition 2.5.*

Proof: It is clear that if T is E -special in the sense of Definition 2.1 then T is E -special in the sense of Definition 2.4. Suppose, therefore, that T is E -special in the sense of Definition 2.4. Fix f mapping T_E into \mathbf{Q} as in Definition 2.4. Let p be a one-to-one mapping from $\mathbf{Q} \times \omega$ into ω . For each $r \in \mathbf{Q}$ let \mathcal{I}_r be the set of all minimal elements of $f^{-1}(r)$. Using the fact that T is Aronszajn, for each $x \in \mathcal{I}_r$ we may let $\langle t_{r,x,k} : k \in \omega \rangle$ enumerate $\{y \in T_E : x \leq y \text{ and } f(y) = r\}$. For

every $z \in T_E$ let $h(z) = p(r, k)$ for the unique r and x and k such that $z = t_{r,x,k}$. It is clear that h maps T_E into ω and whenever $x < y$ are in T_E then $h(x) \neq h(y)$. By Lemma 2.7 we have that T is E -special in the sense of Definition 2.1.

Now suppose that $S \subseteq \omega_1$ consists of limit ordinals. Clearly if T is S -*-special in the sense of Definition 2.2 then T is S -*-special in the sense of Definition 2.5. So, suppose that f is a function that S -*-specializes T in the sense of Definition 2.5. By Lemma 2.6, we may assume that for every $\alpha \in S$ we have that $\alpha = \alpha^\omega$ (ordinal arithmetic). For each $\gamma \in \omega_1$ let \mathcal{I}_γ be the set of minimal elements of $f^{-1}(\gamma)$. For each $\gamma < \omega_1$ and $x \in \mathcal{I}_\gamma$ let $\langle t_{\gamma,x,m} : m \in \omega \rangle$ enumerate $\{y \in T_S : x \leq y \text{ and } f(y) = \gamma\}$. For $\gamma \in \omega_1$ and $x \in \mathcal{I}_\gamma$ and $m \in \omega$, set

$$g(t_{\gamma,x,m}) = \omega \cdot \gamma + m$$

Using the fact that $\text{rk}(t_{\gamma,x,m}) \geq \text{rk}(x) > \gamma$ and $\text{rk}(t_{\gamma,x,m}) \in S$ we have that $\text{rk}(t_{\gamma,i,m}) \geq \gamma^\omega > g(t_{\gamma,x,m})$. It is straightforward to check that g is a function that S -*-specializes T in the sense of Definition 2.2.

The Lemma is established.

Lemma 2.9. *Suppose T is an ω_1 -tree and either T is E -special for some unbounded $E \subseteq \omega_1$ or T is S -*-special for some stationary S consisting of limit ordinals. Then T is neither Suslin nor Kurepa.*

Proof: By Lemmas 2.3 and 2.8 we have that T is not Suslin.

Suppose that T is E -special. Let f be a specializing function mapping T_E into \mathbf{Q} . For every uncountable branch b , choose $r_b \in \mathbf{Q}$ such that $\{y \in b \cap T_E : f(y) = r_b\}$ is uncountable, and let $t_b = \min\{y \in b \cap T_E : f(y) = r_b\}$. The function taking b to t_b is a one-to-one mapping from the set of uncountable branches into T . Hence the number of uncountable branches is at most \aleph_1 .

Now assume that S is a stationary set of countable limit ordinals and T is S -*-special, and assume that f is a function which S -*-specializes T . For every uncountable branch $b \subseteq T$, use Fodor's Theorem to choose $\gamma_b \in \omega_1$ such that $\{y \in b \cap T_S : f(y) = \gamma_b\}$ is uncountable, and let $t_b = \min\{y \in b \cap T_S : f(y) = \gamma_b\}$. The function taking b to t_b is a one-to-one mapping from the set of uncountable branches into T . Hence the number of uncountable branches is at most \aleph_1 . The Lemma is established.

3 (T, S) -#-preserving forcings

In this section we introduce the preservation property that will be used in the main constructions, and we establish some technical properties.

Definition 3.1. Suppose T is an ω_1 -tree and λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ and $T \in N$ and $x \in T$. We say that x is (T, N) -#-generic iff $\text{rk}(x) = \omega_1 \cap N$ and for every $A \in N$ such that $A \subseteq T$ we have $(\exists y < x)(y \in A \text{ or } (\forall z \geq y)(z \notin A))$.

Definition 3.2. Suppose P is a forcing and T is an ω_1 -tree and $S \subseteq \omega_1$ and λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ and $\{P, T, S\} \in N$ and $q \in P$. We say that q is (N, P, S, T) -#-preserving iff q is (N, P) -generic and either $\omega_1 \cap N \in S$ or for every $x \in T$ such that x is (T, N) -#-generic and every P -name A from N such that $\mathbf{1} \Vdash "A \subseteq T,"$ we have that $q \Vdash "(\exists y < x)(y \in A \text{ or } (\forall z \geq y)(z \notin A))."$

Lemma 3.3. Whenever p is (N, P, S, T) -#-preserving and $q \leq p$, then q is (N, P, S, T) -#-preserving. Also, whenever p is (N, P, S, T) -#-preserving and $S \subseteq S'$ then p is (N, P, S', T) -#-preserving.

Proof: Obvious.

Lemma 3.4. Suppose q is (N, P, S, T) -#-preserving and $\omega_1 \cap N \notin S$ and x is (T, N) -#-generic. Then $q \Vdash "x \text{ is } (T, N[G_P])\text{-#-generic}."$

Proof: Necessarily q is N -generic, so $q \Vdash " \text{rk}(x) = \omega_1 \cap N = \omega_1 \cap N[G_P] "$ and T is an ω_1 -tree." Now suppose that $q' \leq q$ and $q' \Vdash "A \in N[G_P] \text{ and } A \subseteq T \text{ and } (\forall y < x)(y \notin A)."$ Because $q' \Vdash "A \in N[G_P]"$ we may take $r \leq q'$ and A' a P -name in N such that $r \Vdash "A' = A."$ We may replace A' by the P -name A^* in N characterized by $\mathbf{1} \Vdash "A^* = A' \text{ if } A' \subseteq T \text{ and } A^* = \emptyset \text{ otherwise}."$ Because r is (N, P, S, T) -#-preserving, we have $r \Vdash "(\exists y < x)(\forall z \geq y)(z \notin A^*)." The Lemma is established.$

Lemma 3.5. Suppose p is (N, P, S, T) -#-preserving and $p \Vdash " \dot{q} \text{ is } (N[G_P], \dot{Q}, S, T)\text{-#-preserving}."$ Then (p, \dot{q}) is $(N, P * \dot{Q}, S, T)$ -#-preserving.

Proof: If $\omega_1 \cap N \in S$, then the Lemma follows from the well-known fact that if p is N -generic and $p \Vdash " \dot{q} \text{ is } N[G_P]\text{-generic},"$ then (p, \dot{q}) is N -generic. So suppose that $\omega_1 \cap N \notin S$ and $A \in N$ is a $P * \dot{Q}$ -name for a subset of T and x

is (T, N) - $\#$ -generic and $(p_1, \dot{q}_1) \leq (p, \dot{q})$ and $(p_1, \dot{q}_1) \Vdash “(\forall y < x)(y \notin A).”$ Fix $\tilde{A} \in N$ a P -name such that $\mathbf{1} \Vdash_P “\tilde{A}$ is a \dot{Q} -name and $\mathbf{1} \Vdash_{\dot{Q}} ‘\tilde{A} = A.’”$ Because p is N -generic, we have that $p \Vdash “T$ is an ω_1 -tree.”

By Lemmas 3.3 and 3.4, we have $p_1 \Vdash “x$ is $(T, N[G_P])$ - $\#$ -generic.” Because p_1 is N -generic we also have that $p_1 \Vdash “\omega_1 \cap N[G_P] = \omega_1 \cap N \notin S.”$ Hence using the fact that $p_1 \Vdash “\dot{q}_1$ is $(N[G_P], \dot{Q}, S, T)$ - $\#$ -preserving and $\dot{q}_1 \Vdash “(\forall y < x)(y \notin \tilde{A}),”$ we have that $p_1 \Vdash “\dot{q}_1 \Vdash “(\exists y < x)(\forall z \geq y)(z \notin \tilde{A}).”$ So there is $(p_2, \dot{q}_2) \leq (p_1, \dot{q}_1)$ and $y < x$ such that $p_2 \Vdash “\dot{q}_2 \Vdash “(\forall z \geq y)(z \notin \tilde{A}).”$ We have $(p_2, \dot{q}_2) \Vdash “(\forall z \geq y)(z \notin A).”$ The Lemma is established.

The following is the key Definition of this paper, in which we isolate the preservation property that we use to maintain non- E -specialness of an appropriately chosen Suslin tree of the ground model. This Definition is analogous to [PIF, Definition IX.4.5], [JSL, Definition 5].

Definition 3.6. Suppose T is an ω_1 -tree and $S \subseteq \omega_1$ and P is a poset. We say that P is (T, S) - $\#$ -preserving iff whenever λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ and $\{T, S, P\} \in N$ and $p \in P \cap N$ then there is $q \leq p$ such that q is (N, P, S, T) - $\#$ -preserving.

In the following Definition, we specify three different ways of collapsing a stationary co-stationary subset of ω_1 . These are well-known (although the third poset is less well-known than it deserves to be).

Definition 3.7. Suppose $S \subseteq \omega_1$. The poset $CU(S)$ is the set of closed, bounded subsets of S ordered by reverse end-extension. The poset $CU^*(S)$ consists of pairs $\langle \sigma, C \rangle$ such that σ is a countable closed subset of S and C is a closed unbounded subset of ω_1 , ordered by $\langle \sigma_1, C_1 \rangle \leq \langle \sigma_2, C_2 \rangle$ iff σ_1 end-extends σ_2 and $C_1 \subseteq C_2$ and $\sigma_1 \subseteq \sigma_2 \cup C_2$. The poset $CU^{**}(S)$ consists of all finite sets F of intervals $[\alpha, \beta]$ such that the elements of F are disjoint, and for every $[\alpha, \beta] \in F$ we have that α is either a successor ordinal or zero or an element of S , ordered by $F_1 \leq F_2$ iff $F_1 \supseteq F_2$.

Lemma 3.8. Suppose $S \subseteq \omega_1$. Suppose P is one of $CU(S)$ or $CU^*(S)$ or $CU^{**}(S)$. Then in $V[G_P]$ we have that $\omega_1 - S$ is non-stationary, and if S is stationary then ω_1 is preserved (in fact, P is S -proper).

Proof: The only possibly unclear case is handled by the observation that if $P = CU^{**}(S)$ then in $V[G_P]$, we have that $\{\alpha : (\exists F \in G_P)(\exists \beta < \omega_1)([\alpha, \beta] \in F$

and α is a limit ordinal) $\}$ is a closed unbounded subset of S . The S -properness of $CU^{**}(S)$ is demonstrated in the proof of Lemma 3.9.

The “case 2” part of the proof of Lemma 3.9 recalls the proof of [Sh, Lemma IX.4.6]. Lemma 3.9 is analogous to [PIF, XXX] and [JSL, Lemma 20].

Lemma 3.9. *Suppose $P = CU(S)$ or $P = CU^*(S)$ or $P = CU^{**}(S)$ and λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ and $\{T, S, P\} \in N$ and $\omega_1 \cap N \in S$ and $p \in P \cap N$. Then there is $q \leq p$ such that q is $(N, P, \omega_1 - S, T)$ -#-preserving.*

Proof: Let $\delta = \omega_1 \cap N$.

Case 1: $P = CU(S)$ or $P = CU^*(S)$.

Let $\langle (x_n, A_n) : n \in \omega \rangle$ enumerate the set of all $\langle x, A \rangle$ such that $A \in N$ is a P -name for a subset of T and x is (T, N) -#-generic (if there are no such x then ignore requirement (3) below). Let $\langle D_n : n \in \omega \rangle$ list the set of all $D \in N$ such that $D \subseteq P$ is open dense.

Claim: There is a sequence $\langle p_n : n \in \omega \rangle$ such that $p_0 = p$ and for all $n \in \omega$ we have that each of the following holds:

- (1) $p_{n+1} \leq p_n$
- (2) $p_{n+1} \in D_n \cap N$
- (3) either $p_n \Vdash “(\exists y < x_n)(\forall z \geq y)(z \notin A_n)”$ or for some $y < x_n$ we have $p_{n+1} \Vdash “y \in A_n.”$

Proof of Claim: Given p_n , take $p'_n \leq p_n$ such that $p'_n \in D_n \cap N$. Let $Y = \{y \in T : p'_n \Vdash “y \notin A_n”\}$.

If $(\exists y < x_n)(y \in Y)$ then we may take $p_{n+1} \leq p'_n$ and $y < x_n$ such that $p_{n+1} \Vdash “y \in A_n.”$ We may assume $p_{n+1} \in N$, and hence the second disjunct in requirement (3) holds.

If instead $(\forall y < x_n)(y \notin Y)$, then because x_n is (T, N) -#-generic we have that there is some $y < x_n$ such that $(\forall z \geq y)(z \notin Y)$. Hence the first disjunct of requirement (3) holds.

If $P = CU(S)$ then let $q = \bigcup \{p_n : n \in \omega\} \cup \{\delta\}$, whereas if $P = CU^*(S)$ then let $q = \langle \bigcup \{\sigma_n : n \in \omega\} \cup \{\delta\}, \bigcap \{C_n : n \in \omega\} \rangle$ where $p_n = \langle \sigma_n, C_n \rangle$ for every $n \in \omega$. Because $\delta \in S$ we have $q \in P$. Clearly q is as required.

Case 2: $P = CU^{**}(S)$.

Let $\delta^* = \sup\{f(\delta)+1 : f \in N \text{ is a function and } f(\delta) \in \omega_1\}$. Let $q = p \cup \{\delta', \delta'\}$, where $\delta' \geq \delta^*$ is not a limit ordinal outside of S (hence $q \in P$). We show that q

is $(N, P, \omega_1 - S, T)$ -#-preserving.

First we show that q is (N, P) -generic. Given $D \in N$ a dense open subset of P , and given $q^* \leq q$, we find $r \leq q^*$ such that r is below some element of $D \cap N$. Choose $r' \leq q^*$ such that $r' \in D$. We have

$$N \models “(\exists p^* \leq (r' \cap N))(p^* \in D)”$$

Choose $p^* \in N$ to be a witness. Set $r = r' \cup p^*$. Clearly $r \in P$ and r is as required.

Now suppose, towards a contradiction, that x is (T, N) -#-generic and $A \in N$ is a P -name for a subset of T , and $q' \leq q$ and $q' \Vdash “(\forall y < x)(y \notin A \text{ and } (\exists z \geq y)(z \in A))”$.

Let $\alpha = \sup(\bigcup(q' \cap N))$. In other words, the “largest” interval in $q' \cap N$ is $[\gamma, \alpha]$ for some γ .

For p_1 and p_2 in P , define $p_1 \leq^* p_2$ iff there is some β such that $p_2 = \{[\eta, \gamma] \in p_1 : \gamma \leq \beta\}$. Essentially, $p_1 \leq^* p_2$ iff p_1 “end-extends” p_2 .

Let $R = \{y \in T : \text{rk}(y) > \alpha\}$ and for all $y \in R$ let $J(y) = \{\gamma < \omega_1 : (\exists \alpha^* < \omega_1)(\gamma \leq \alpha^* \text{ and } \gamma \text{ is not a limit ordinal outside of } S \text{ and } (\exists \tilde{q} \leq^* (q' \cap N \cup \{[\gamma, \alpha^*]\}))(\tilde{q} \Vdash “y \notin A”))\}$. Let F be the function with domain equal to $\{y \in R : J(y) \neq \emptyset\}$ characterized by $(\forall y \in \text{dom}(F))(F(y) = \sup(J(y)))$. Let $A^* = \{y \in R : J(y) \neq \emptyset \text{ and } F(y) = \omega_1\}$.

Because x is (T, N) -#-generic, we may fix $y < x$ such that either $y \notin A^*$ or $(\forall z \geq y)(z \in A^*)$.

Case 1. $y \notin A^*$

Claim: $(q' \cap N) \nVdash “y \notin A.”$

Suppose instead that $(q' \cap N) \Vdash “y \notin A.”$ We have $q' \cap N \cup \{[\gamma + 1, \gamma + 1]\}$ witnesses $\gamma + 1 \in J(y)$ for every countable $\gamma \geq \alpha$, hence $F(y) = \omega_1$, contradicting the fact that $y \notin A^*$. The Claim is established.

By the Claim we may take $q^+ \leq (q' \cap N)$ such that $q^+ \Vdash “y \in A”$ and $q^+ \in N$. Clearly we have that $(q^+ \cup q') \in P$. But $q^+ \Vdash “y \in A”$ and $q' \Vdash “y \notin A.”$ This is impossible.

Case 2. $(\forall z \geq y)(z \in A^*)$

We have $q' \Vdash “(\exists z \geq y)(z \in A).”$ Choose $q^+ \leq q'$ and $z \geq y$ such that $q^+ \Vdash “z \in A.”$ Fix γ a countable ordinal greater than $\sup(\bigcup q^+)$. Because $z \in A^*$ we know that $J(z)$ is not empty. Furthermore, $F(z) = \omega_1$, so we may take $\gamma^* \in J(z)$ such that $\gamma \leq \gamma^*$. We may $\alpha^* \geq \gamma^*$ and $\tilde{q} \leq^* (q' \cap N \cup \{[\gamma^*, \alpha^*]\})$

such that $\tilde{q} \Vdash "z \notin A."$ Clearly $(q^+ \cup \tilde{q}) \in P$. We have $(q^+ \cup \tilde{q}) \Vdash "z \in A \text{ and } z \notin A."$ This is impossible, hence the Lemma is established.

The following Lemma is analogous to [JSL, Lemma 21].

Lemma 3.10. *Suppose P is a poset and T is an ω_1 -tree and $S \subseteq \omega_1$. Suppose λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ containing $\{P, T, S\}$. Suppose $p \in P$ is (N, P, S, T) - $\#$ -preserving and A is a P -name in N for a Q -name in $N[G_P]$ that names a subset of T and $p \Vdash "\dot{Q}$ is (T, S) - $\#$ -preserving and $\dot{Q} \in N[G_P]$ and $q \in \dot{Q} \cap N[G_P]"$ and $x \in T$ and x is (T, N) - $\#$ -generic and $\omega_1 \cap N \notin S$. Then there is a P -name r such that $p \Vdash "r \leq q$ and $r \in N[G_P]"$ and $(p, r) \Vdash "(\exists y < x)(y \in A \text{ or } (\forall z \geq y)(z \notin A))."$*

Proof: Let $D = \{p' \leq p : p' \Vdash "(\exists y < x)(q \Vdash '(\forall z \geq y)(z \notin A)')"$ or $p' \Vdash "(\exists y < x)(\exists q' \leq q)(q' \in N[G_P] \text{ and } q' \Vdash 'y \in A')"$ }.

Claim 1. D is dense below p .

Proof: Suppose $p^+ \leq p$. Because $p \Vdash "q \in N[G_P]"$, we may take $\tilde{p} \leq p^+$ and q^* a P -name in N such that $\tilde{p} \Vdash "q^* = q."$ Take B to be a P -name in N characterized by $\mathbf{1} \Vdash "B = \{y \in T : q^* \Vdash 'y \notin A'\}."$

By Lemma 3.4 we have $\tilde{p} \Vdash "x \text{ is } (T, N[G_P])\text{-}\# \text{-generic}."$ and therefore we can take $p_1 \leq \tilde{p}$ and $y < x$ such that either $p_1 \Vdash "y \in B"$ or $p_1 \Vdash "(\forall z \geq y)(z \notin B)."$

If we have $p_1 \Vdash "y \in B,"$ then p_1 witnesses the second disjunct in the definition of D and we are done. If instead $p_1 \Vdash "(\forall z \geq y)(z \notin B),"$ we have $p_1 \Vdash "(\forall z \geq y)(q^* \Vdash 'z \notin A'),"$ and thus p_1 witnesses the first disjunct of the definition of D . In either case, the Claim is established.

We now define a function f with domain D as follows. If $p' \in D$ and $p' \Vdash "(\exists y < x)(q \Vdash '(\forall z \geq y)(z \notin A)')"$, then we let $f(p') = q$. If instead $p' \Vdash "(\exists q' \leq q)(\exists y < x)(q' \in N[G_P] \text{ and } q' \Vdash 'y \in A')"$ then we choose some such q' , and set $f(p') = q'$. Let \mathcal{J} be a maximal antichain of P such that $\mathcal{J} \subseteq D$. Let r be a P -name such that for every $p' \in \mathcal{J}$ we have $p' \Vdash "r = f(p')."$

By Claim 1 we clearly have that $p \Vdash "r \in N[G_P] \text{ and } r \leq q."$

Claim 2. $p \Vdash "r \Vdash '(\exists y < x)(y \in A \text{ or } (\forall z \geq y)(z \notin A)).'"$

Proof: Suppose $p_1 \leq p$. Take $p' \in \mathcal{J}$ and $p_2 \leq p_1$ such that $p_2 \leq p'$.

Case 1: $p' \Vdash "(\exists y < x)(q \Vdash '(\forall z \geq y)(z \notin A)')"$

Clearly $p' \Vdash "r \Vdash '(\exists y < x)(\forall z \geq y)(z \notin A).'"$

Case 2: Otherwise.

Because Case 1 fails and $p' \in D$ we have by choice of r that $p' \Vdash “(\exists y < x)$
 $(r \Vdash ‘y \in A’).”$

In either case, we have $p_2 \Vdash “r \Vdash ‘(\exists y < x)(y \in A \text{ or } (\forall z \geq y)(z \notin A))’”$

The Claim is established, and the Lemma is proved.

Lemma 3.11. *Suppose x is (T, N) -#-generic and $A \in N$ and A is an antichain of T and $T \in N$. Then $x \notin A$.*

Proof: Suppose $x \in A$. Then $(\forall y < x)(y \notin A)$. Hence $(\exists y < x)(\forall z \geq y)$
 $(z \notin A)$. Hence $x \notin A$. The Lemma is established.

4 Sharply Suslin trees

Definition 4.1. Suppose T is a Suslin tree. We say that T is sharply Suslin iff for every sufficiently large regular cardinal λ and every countable elementary substructure M of H_{λ^+} we have that there is a closed unbounded $C \subseteq \omega_1$ such that for every $N \in M$ such that N is a countable elementary substructure of H_λ containing T and $\omega_1 \cap N \in C$, and every $x \in T_{\omega_1 \cap N}$, we have that x is (T, N) -#-generic.

Definition 4.2. \diamond^* is the following principle: there is a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ such that S_α is a countable subset of $\mathcal{P}(\alpha)$ and for every $X \subseteq \omega_1$ there is a closed unbounded $C \subseteq \omega_1$ such that for every $\alpha \in C$ we have $X \cap \alpha \in S_\alpha$.

Lemma 4.3. Suppose $V = L$. Then \diamond^* holds.

Proof: See [Devlin, Theorem III.3.5].

Lemma 4.4. Suppose \diamond^* holds. Then there is a sharply Suslin tree.

Proof: Let $\langle S_\alpha : \alpha < \omega_1 \rangle$ be a \diamond^* -sequence. Because \diamond^* implies \diamond , we may also fix a \diamond -sequence $\langle Z_\alpha : \alpha < \omega_1 \rangle$.

Given λ and M as in Definition 4.1, let $X \subseteq \omega_1$ code $\{x \cap \omega_1 : x \in M\}$. For example, we may let $\langle \theta_i : i \in \omega \rangle$ list M , and let $X = \{\omega\alpha + i : \alpha \in \theta_i\}$. Let C be a closed unbounded subset of $\{\alpha < \omega_1 : X \cap \alpha \in S_\alpha \text{ and } \alpha \text{ is indecomposable}\}$. Build T recursively such that whenever $\beta < \omega_1$ is an indecomposable ordinal then we have $T_{<\beta} = \beta$ and we build T_β as follows.

Let $\langle \beta_n : n \in \omega \rangle$ be an increasing sequence of ordinals cofinal in β . Let $\langle B_i^\beta : i \in \omega \rangle$ list S_β . Let $\langle A_i^\beta : i \in \omega \rangle$ list $\{\{\alpha < \beta : \omega\alpha + i \in B_k^\beta\} : i \in \omega \text{ and } k \in \omega\}$.

Thus for each $x \in M$, if $\beta \in C$ then we have that $x \cap \beta$ is equal to A_i^β for some $i \in \omega$.

Build $\langle (A')_i^\beta : i \in \omega \rangle$ such that for every $n \in \omega$ we have each of the following:

- (1) $(A')_n^\beta$ is an antichain of $T_{<\beta}$,
- (2) $(A')_n^\beta$ is predense above A_n^β , i.e., $(\forall x \in A_n^\beta)(\forall y \geq x)(\exists z \in (A')_n^\beta)(z \text{ is comparable with } y)$,
- (3) for every $i \leq n$ and every $y' \in (A')_i^\beta$ and every $y \in (A')_n^\beta$ we have either $y' \leq y$ or y' is incomparable with y ,
- (4) for every $y \in (A')_n^\beta$ we have $\text{rk}(y) \geq \beta_n$.

There is no problem in doing this.

Now construct T_β such that

- (1) for every $x \in T_\beta$ and every $n \in \omega$ there is $y < x$ such that either $y \in (A')_n^\beta$ or $(\forall z \geq y)(z \notin (A')_n^\beta)$, and
- (2) for every $y \in T_{<\beta}$ there is $x \in T_\beta$ such that $y < x$,
- (3) for every $x \in T_\beta$ there is $y < x$ such that either $y \in Z_\beta$ or $(\forall z \geq y)(z \notin Z_\beta)$.

There is no problem in this.

It is easy to see that for every $x \in T_\beta$ and every $n \in \omega$ there is $y < x$ such that either $y \in A_n^\beta$ or $(\forall z \geq y)(z \notin A_n^\beta)$. Also it is easy to see that the tree T that is constructed in this way is a Suslin tree.

It is easy to see that C is the required witness to the assertion that λ and M do not constitute a counterexample to the fact that T is sharply Suslin.

The Lemma is established.

Lemma 4.5. *Suppose T is a sharply Suslin tree and $S \subseteq \omega_1$ is co-stationary and P is (T, S) -#-preserving and $\mathbf{1} \Vdash$ “ T is Aronszajn.” Then $\mathbf{1} \Vdash$ “ $(\forall E \subseteq \omega_1 \text{ unbounded})(T \text{ is not } E\text{-special})$.”*

Proof: Suppose, towards a contradiction, that E and f are P -names and $p \in P$ and $p \Vdash$ “ $E \subseteq \omega_1$ is unbounded and f is an E -specializing function for T in the sense of Definition 2.1.” Take λ a large enough regular cardinal and M a countable elementary substructure of H_{λ^+} containing $\{T, S, P, E, f, p\}$. Take C as in Definition 4.1 and fix $N \in M$ such that N is a countable elementary substructure of H_λ containing $\{T, S, P, E, f, p\}$ and such that $\omega_1 \cap N \in C$ and $\omega_1 \cap N \notin S$. Fix $x \in T_{\omega_1 \cap N}$. Take $q \leq p$ such that q is (N, P, S, T) -#-preserving and, by a further strengthening of q , we may take $r \in \mathbf{Q}$ such that $q \Vdash$ “ $f(z) = r$ for some $z \geq x$.” Necessarily we have $q \Vdash$ “ $(\forall y < x)(y \notin f^{-1}(r))$.” Hence

$q \Vdash “(\exists y < x)(\forall y' \geq y)(y' \notin f^{-1}(r)).”$ This contradicts the fact that $q \Vdash “z \in f^{-1}(r).”$ The Lemma is established.

5 Iteration of (T, S) -#-preserving forcings

In this section we show that the property “ (T, S) -#-preserving” is preserved by countable support forcing iteration (and a bit more). This is a variant of [PIF XXXX], [JSL, Definition 22 and Lemmas 23 and 24].

Definition 5.1. Suppose $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support iteration of forcing. We say that P_κ is strictly (T, S) -#-preserving iff whenever λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ and $\{T, S, \langle P_\eta : \eta \leq \kappa \rangle\} \in N$ and $\alpha \in \kappa \cap N$ and $q \in P_\alpha$ is (N, P_α, S, T) -#-preserving and $q \Vdash “\dot{p} \in \dot{P}_{\alpha, \kappa} \cap N[G_{P_\alpha}],”$ then there is $r \in P_\kappa$ such that $r \restriction \alpha = q$ and $q \Vdash “r \restriction [\alpha, \kappa] \leq \dot{p}”$ and r is (N, P_κ, S, T) -#-preserving and $\text{supt}(r) \subseteq \alpha \cup N$.

Lemma 5.2. Suppose P_κ is strictly (T, S) -#-preserving. Then P_κ is (T, S) -#-preserving.

Proof: Take $\alpha = 0$ in Definition 5.1.

Lemma 5.3. The following are equivalent:

- (1) P_κ is strictly (T, S) -#-preserving,
- (2) For some regular $\lambda > \omega_1$ such that $P_\kappa \in H_\lambda$ there is a closed unbounded $C \subseteq [H_\lambda]^\omega$ such that whenever $N \in C$ and $\eta \in \kappa \cap N$ and q is (N, P_η, S, T) -#-preserving and $q \Vdash “\dot{p} \in \dot{P}_{\eta, \kappa} \cap N[G_{P_\eta}]”$ then there is $r \in P_\kappa$ such that $r \restriction \eta = q$ and $q \Vdash “r \restriction [\eta, \kappa] \leq \dot{p}”$ and r is (N, P_κ, S, T) -#-preserving and $\text{supt}(r) \subseteq \eta \cup N$,
- (3) For some regular $\lambda > 2^{\aleph_1}$ such that the power set of P_κ is an element of H_λ we have that whenever N is a countable elementary substructure of H_λ and $\{P_\kappa, T, S\} \in N$ and $\eta \in \kappa \cap N$ and q is (N, P_η, S, T) -#-preserving and $q \Vdash “\dot{p} \in \dot{P}_{\eta, \kappa} \cap N[G_{P_\eta}]”$ then there is $r \in P_\kappa$ such that $r \restriction \eta = q$ and $q \Vdash “r \restriction [\eta, \kappa] \leq \dot{p}”$ and r is (N, P_κ, S, T) -#-preserving and $\text{supt}(r) \subseteq \eta \cup N$,
- (4) For every regular $\lambda > \omega_1$ such that $P_\kappa \in H_\lambda$ there is a closed unbounded $C \subseteq [H_\lambda]^\omega$ such that whenever $N \in C$ and $\eta \in \kappa \cap N$ and q is (N, P_η, S, T) -#-preserving and $q \Vdash “\dot{p} \in \dot{P}_{\eta, \kappa} \cap N[G_{P_\eta}]”$ then there is $r \in P_\kappa$ such that $r \restriction \eta = q$ and $q \Vdash “r \restriction [\eta, \kappa] \leq \dot{p}”$ and r is (N, P_κ, S, T) -#-preserving and $\text{supt}(r) \subseteq \eta \cup N$.

Proof: (1) trivially implies (2) and (3), and (4) trivially implies (2). We show (2) implies (1). Fix λ to be the least witness to (2), and suppose $\mu > 2^{\aleph_1}$ is a regular cardinal such that the power set of P_κ is in H_μ . Let $C_0 = \{M \in [H_\lambda]^\omega : \{P_\kappa, T, S\} \in M \text{ and whenever } \eta \in \kappa \cap M \text{ and } q \text{ is } (M, P_\eta, S, T)\text{-}\# \text{-preserving and } q \Vdash \text{“}\dot{p} \in \dot{P}_{\eta, \kappa} \cap M[G_{P_\eta}]\text{” then there is } r \in P_\kappa \text{ such that } r \restriction \eta = q \text{ and } q \Vdash \text{“}r \restriction [\eta, \kappa] \leq \dot{p}\text{” and } r \text{ is } (M, P_\kappa, S, T)\text{-}\# \text{-preserving and } \text{supt}(r) \subseteq \eta \cup M\}\}$. Suppose N is a countable elementary substructure of H_μ and $\{P_\kappa, T, S\} \in N$. Then $C_0 \in N$ and $\lambda \in N$, because C_0 and λ are Δ_1 -definable from the parameters $P_\kappa, \mathcal{P}(P_\kappa), \mathcal{P}(\omega_1), T$, and S . Take $C \subseteq C_0$ such that C is a closed unbounded subset of $[H_\lambda]^\omega$ and $C \in N$. Let $\langle \theta_n : n \in \omega \rangle$ enumerate $N \cap H_\lambda$. By recursion, build $\langle M_n : n \in \omega \rangle$ such that for every $n \in \omega$ we have M_n is a countable elementary substructure of $N \cap H_\lambda$ and $\{M_n, \theta_n\} \in M_{n+1}$ and $M_n \in C$. We have therefore that $N \cap H_\lambda = \bigcup \{M_n : n \in \omega\} \in C$. Therefore, whenever $\eta \in \kappa \cap N$ and q is $(N, P_\eta, S, T)\text{-}\# \text{-preserving}$ and $q \Vdash \text{“}\dot{p} \in \dot{P}_{\eta, \kappa} \cap N[G_{P_\eta}]\text{”}$, then clearly q is $(N \cap H_\lambda, P_\eta, S, T)\text{-}\# \text{-preserving}$ and $q \Vdash \text{“}\dot{p} \in (N \cap H_\lambda)[G_{P_\eta}]\text{”}$, and therefore there is $r \in P_\kappa$ such that $r \restriction \eta = q$ and $q \Vdash \text{“}r \restriction [\eta, \kappa] \leq \dot{p}\text{”}$ and r is $(N \cap H_\lambda, P_\kappa, S, T)\text{-}\# \text{-preserving}$ and $\text{supt}(r) \subseteq \eta \cup N$. Clearly r is $(N, P_\kappa, S, T)\text{-}\# \text{-preserving}$. This verifies that (1) holds.

We now show that (3) implies (4). Given λ as in (4), let $C = \{M \in [H_\lambda]^\omega : M \text{ is a countable elementary substructure of } H_\lambda \text{ and } \{P_\kappa, T, S\} \in M \text{ and there is some regular } \mu > 2^{\aleph_1} \text{ and } N \text{ a countable elementary substructure of } H_\mu \text{ such that } 2^{\omega_1} \in N \text{ and } \mathcal{P}(P_\kappa) \in N \text{ and } M = N \cap H_\lambda\}\}$. Then C witnesses that (4) holds. The Lemma is established.

Theorem 5.4. *Suppose T is sharply Suslin and $S \subseteq \omega_1$ and $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle \dot{Q}_\eta : \eta < \kappa \rangle$. Suppose for every $\eta < \kappa$ we have that \mathcal{I}_η and S_η are P_η -names. Suppose for every $\eta < \kappa$ we have either*

- (1) $\mathbf{1} \Vdash_{P_\eta} \text{“}\dot{Q}_\eta \text{ is } (T, S)\text{-}\# \text{-preserving,“}$ or
- (2) $\mathbf{1} \Vdash_{P_\eta} \text{“}\mathcal{I}_\eta \text{ is an antichain of } T \text{ and } S_\eta = \{\text{rk}(x) : x \in \mathcal{I}_\eta\} \text{ and } \dot{Q}_\eta \text{ is one of } CU(S \cup (\omega_1 - S_\eta)) \text{ or } CU^*(S \cup (\omega_1 - S_\eta)) \text{ or } CU^{**}(S \cup (\omega_1 - S_\eta))\text{”}$

Then $\langle P_\eta : \eta \leq \kappa \rangle$ is strictly $(T, S)\text{-}\# \text{-preserving}$.

Comment: Note that the stationary sets which are collapsed in case (2) are not in the ground model. Indeed, we have that P_α is proper for $\alpha \leq \kappa$.

Proof: We prove by induction on κ that whenever λ is a sufficiently large regu-

lar cardinal and N is a countable elementary substructure of H_λ and $\{(\mathcal{I}_\eta, S_\eta) : \eta < \kappa\}, P_\kappa, T, S\} \in N$ and $\alpha \in \kappa \cap N$ and q is (N, P_α, S, T) -#-preserving and $q \Vdash \dot{p} \in \dot{P}_{\alpha, \kappa} \cap N[G_{P_\alpha}]$ then there is $r \in P_\kappa$ such that $r \restriction \alpha = q$ and $q \Vdash \text{"}r \restriction [\alpha, \kappa] \leq \dot{p}\text{"}$ and $\text{supt}(r) \subseteq \alpha \cup N$ and r is (N, P_κ, S, T) -#-preserving. This differs from the definition of (T, S) -#-preserving insofar as we assume that N must contain a certain additional parameter (namely, $\langle (\mathcal{I}_\eta, S_\eta) : \eta < \kappa \rangle$), but by Lemma 5.3 this is immaterial.

In line with the induction on κ , we assume that P_β is strictly (T, S) -#-preserving for every $\beta < \kappa$. Assume λ and N are given.

Case 1 (successor step). Suppose $\kappa = \eta + 1$.

Necessarily $\eta \in N$, so we may assume that $\eta = \alpha$. If $\mathbf{1} \Vdash_{P_\eta} \dot{Q}_\eta$ is (T, S) -#-preserving," then we are done by Lemma 3.5. Otherwise, by Lemmas 3.4 and 3.11 we have that $q \Vdash \text{"}\omega_1 \cap N[G_{P_\eta}] = \omega_1 \cap N \notin S_\eta\text{"}$, so by Lemmas 3.5 and 3.9 we are again done.

Case 2 (limit step). Suppose κ is a limit ordinal.

Let $\kappa' = \sup(\kappa \cap N)$, and let $\langle \alpha_i : i \in \omega \rangle$ be a strictly increasing sequence of ordinals from $\kappa \cap N$ cofinal in κ' such that $\alpha_0 = \alpha$. Let $\langle \sigma_i : i \in \omega \rangle$ list all P_κ -names σ in N such that $\mathbf{1} \Vdash_{P_\kappa} \text{"}\sigma \text{ is an ordinal.}"$ Let $\langle (x_n, A_n) : n \in \omega \rangle$ list all pairs $\langle x, A \rangle$ such that $A \in N$ is a P_κ -name for a subset of T and x is (T, N) -#-generic. Build a sequence $\langle (q_n, \dot{p}_n) : n \in \omega \rangle$ such that $q_0 = q$ and $\dot{p}_0 = \dot{p}$ and for every $m \in \omega$ we have

- (1) q_m is (N, P_{α_m}, S, T) -#-preserving and $\text{supt}(q_m) \subseteq \alpha \cup N$,
- (2) $q_m \Vdash \text{"}q_{m+1} \restriction [\alpha_m, \alpha_{m+1}] \leq \dot{p}_m \restriction \alpha_{m+1}\text{"}$ and q_{m+1} is $(N, P_{\alpha_{m+1}}, S, T)$ -#-preserving,
- (3) $q_{m+1} \restriction \alpha_m = q_m$,
- (4) $q_{m+1} \Vdash \text{"}\dot{p}_{m+1} \leq \dot{p}_m \restriction [\alpha_{m+1}, \kappa) \text{ and } \dot{p}_{m+1} \in N[G_{P_{\alpha_{m+1}}}] \text{ and } \dot{p}_{m+1} \text{ decides the value of } \sigma_m\text{"}$,
- (5) $q_{m+1} \Vdash \text{"}\dot{p}_{m+1} \Vdash \text{"}(\exists y < x_m)(y \in A_m \text{ or } (\forall z \geq y)(z \notin A_m))\text{"}$ "

This is possible by Lemma 3.10 and the induction hypothesis.

Take $r \in P_\kappa$ such that $\text{supt}(r) \subseteq \alpha \cup N$ and for every $m \in \omega$ we have that $r \restriction \alpha_m = q_m$. This concludes the induction, and thereby establishes the Lemma.

6 The models

Shelah (item (2) below) and, later, Schlindwein, have constructed models of each of the following:

- (1) every Aronszajn tree is S -*-special (S an arbitrary stationary set that is in the ground model) and some Aronszajn tree T is not S' -*-special whenever $S' - S$ is stationary, and CH holds [JSL],
- (2) every Aronszajn tree is S -*-special (S an arbitrary stationary set that is in the ground model) and some Aronszajn tree T is not S' -*-special whenever $S' - S$ is stationary, and CH fails ([Sh], or use [JSL] with $CU^{**}(S \cup (\omega_1 - S_\alpha))$ in place of $CU(S \cup (\omega_1 - S_\alpha))$),
- (3) Suslin's hypothesis plus some Aronszajn tree T has no stationary antichain, plus CH fails [APAL],
- (4) Suslin's hypothesis plus some Aronszajn tree T has no stationary antichain, plus CH holds [APAL2],
- (5) every ω_1 -tree is S -*-special for S an arbitrary stationary set that is in the ground model (in particular, Kurepa's hypothesis fails) and some ω_1 -tree T is not S' -*-special whenever $S' - S$ stationary, plus CH holds [STACY],
- (6) same as (5) but CH fails (use [STACY] but with $CU^{**}(S \cup (\omega_1 - S_\alpha))$ in place of $CU(S \cup (\omega_1 - S_\alpha))$).

Models (5) and (6) require an inaccessible cardinal.

We claim that in variants of each of these six models (seven, actually, as there are two different constructions cited in item (2)) there is no unbounded $E \subseteq \omega_1$ such that T is E -special. The demonstrations are all entirely similar to each other, except in Shelah's construction for item (2), where the demonstration of (S, T) -#-preserving for the forcing of [PIF XXXXX] is similar to the proof of Lemma 3.9 (second case) above. Because the changes to previously published material are easily explained, we do not give complete proofs for all six models.

7 The first two models

In this section, we show that a variation of the model from [JSL] satisfies that there is an Aronszajn tree T^* and a stationary set S^* such that every Aronszajn tree is S^* -*-special, and whenever $S' - S^*$ is stationary then T^* is not S' -*-special, and for every unbounded $E \subseteq \omega_1$ we have that T^* is not E -special. The model

in [JSL] was used to solve the problem of constructing a model of ZFC plus CH plus SH plus not every Aronszajn tree is special (answering a question posed by Shelah [PIF, XXX]).

Throughout this section, we fix S^* a stationary co-stationary subset of ω_1 and we fix an Aronszajn tree T^* . In the end, we will use a sharply Suslin tree in the ground model as T^* , so that the final poset P_{ω_2} will force for every $S' \subseteq \omega_1$ such that $S' - S^*$ is stationary, then T^* is not S' -*-special, and for every unbounded $E \subseteq \omega_1$ we have that T^* is not E -special.

For T an Aronszajn tree, we let T^n be the Aronszajn tree consisting of N -tuples of elements of T , all of which have the same rank. For $x \in T$ and $\beta \leq \text{rk}(x)$ we let $x \restriction \beta$ be the unique $y \leq x$ such that $\text{rk}(y) = \beta$. We turn our attention to defining the posets that will be used as the constituent posets of the iteration.

Definition 7.1. We say that R is a finite rectangle iff there is some $n = n(R) \in \omega$ and some sequence $\langle R_i : i < n \rangle$ such that $R = R_0 \times R_1 \times \cdots \times R_{n-1}$ and for each $i < n$ we have that R_i is a finite subset of ω_1 .

Definition 7.2. Suppose T is an Aronszajn tree and $\gamma < \omega_1$ and $n \in \omega$ and $\bar{x} \in T_\gamma^n$ and f is an ordinal-valued function and $R = R_0 \times R_1 \times \cdots \times R_{n-1}$ is a finite rectangle. Then we define $\heartsuit(\alpha, \bar{x}, f, R)$ to mean that whenever $\alpha < \beta \leq \gamma$ and $i < n$ and $x_i \restriction \beta \in \text{dom}(f)$ then $f(x_i \restriction \beta) \notin R_i$.

Definition 7.3. Suppose T is an Aronszajn tree. We let $P'(T)$ be the poset whose universe is $\{\langle f, S \rangle : S \text{ is a countable set of countable limit ordinals and } f \text{ is an } S\text{-*-specializing function, and } \text{cl}(S) \cap S^* \subseteq S\}$. The ordering is given by co-ordinatewise reverse end-extension.

Definition 7.4. Γ is a T -promise iff there is a closed unbounded $C = C(\Gamma) \subseteq \omega_1$ and an integer $n = n(\Gamma)$ and an $\bar{x} = \min(\Gamma) \in \Gamma$ such that $\Gamma \subseteq T_C^n$ and whenever $\alpha < \beta$ are in C and $\bar{y} \in \Gamma \cap T_\alpha^n$ then there is an infinite $W \subseteq \Gamma \cap T_\beta^n$ such that for every $\bar{z} \in W$ we have $\bar{x} \leq \bar{y} \leq \bar{z}$, and distinct elements of W have disjoint ranges. We also require that for every $\bar{z} \in \Gamma$ and every $\alpha \in \text{rk}(\bar{z}) \cap C$ we have $\bar{z} \restriction \alpha \in \Gamma$.

The following Fact is proved in [PIF XXXX] and [JSL, Lemma 50].

Fact 7.5. Suppose $\Delta \subseteq T^n$ is uncountable and downwards closed and every element of Δ is comparable with $\bar{x} \in T^n$. Then there is some T -promise $\Gamma \subseteq \Delta$

such that $\min(\Gamma) = \bar{x}$.

Definition 7.6. Suppose $\langle f, S \rangle \in P'(T)$ and Γ is a T -promise. We say that $\langle f, S \rangle$ fulfills Γ iff $S - \text{rk}(\min(\Gamma)) \subseteq C(\Gamma)$ and whenever $\beta \in C(\Gamma)$ and $\alpha \in C(\Gamma) \cap S \cap \beta$ and $\bar{y} \in \Gamma \cap T_\beta^n$ and R is a finite rectangle with $n(\Gamma) = n(R)$ then there is an infinite $W \subseteq \Gamma \cap T_\beta^n$ such that distinct elements of W have disjoint ranges and for every $\bar{w} \in W$ we have $\bar{y} \leq \bar{w}$ and $\heartsuit(\alpha, \bar{w}, f, R)$.

Note that in Definition 7.6 we do not assume that $\beta \in S$.

Contrast the following Definition with [JSL, Definition 52]. The difference is that in [JSL], it is required that Ψ be countable.

Definition 7.7. $P(T)$ is the poset whose universe consists of triples $\langle f, S, \Psi \rangle$ such that $\langle f, S \rangle \in P'(T)$ and Ψ is a set of T -promises that $\langle f, S \rangle$ fulfills such that for every $\alpha < \omega_1$ we have that $\{\Gamma \in \Psi : \text{rk}(\min(\Gamma)) < \alpha\}$ is countable. The ordering is given by $\langle f', S', \Psi' \rangle \leq \langle f, S, \Psi \rangle$ iff $f \subseteq f'$ and $S' \cap (\sup(S) + 1) = S$ and $\Psi \subseteq \Psi'$.

For $p \in P(T)$ we will use f_p , S_p , and Ψ_p to denote the components of p , and we set $\text{ht}(p)$ to equal $\sup(S_p)$. We set $C(\Psi_p) = \{\gamma < \omega_1 : (\forall \Gamma \in \Psi_p)(\text{rk}(\min(\Gamma)) > \gamma \text{ or } \gamma \in C(\Gamma))\}$. Note that $C(\Psi_p)$ is closed and unbounded.

The following Lemmas correspond to [JSL, Lemmas 53 through 56]. Although the poset $P(T)$ referred to in [JSL] differs from $P(T)$ in that [JSL] required Ψ_p is countable for every $p \in P(T)$, the difference is immaterial to the proof of these Lemmas.

Lemma 7.8. Suppose $p \in P(T)$ and $\text{ht}(p) = \alpha < \beta \in C(\Psi_p)$. Suppose R is a finite rectangle and $\bar{z} \in T_\beta^n$. Then there is $q \leq p$ such that $\text{ht}(q) = \beta$ and $\heartsuit(\alpha, \bar{z}, f_q, R)$.

Proof: In the proof of [JSL, Lemma 53] simply replace the clause “ $\Gamma \in \Psi_p$ ” with “ $\Gamma \in \Psi_p$ and $\text{rk}(\min(\Gamma)) \leq \beta$.” Besides that change, the proof is unchanged.

The following Lemma is [JSL, Lemma 54] but, as usual, for a slightly different poset. The same Lemma basically appears in [PIF, chapter V] for a different poset.

Lemma 7.9. Suppose that λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ and $P(T) \in N$ and $p \in P(T) \cap N$ and

$D \in N$ is dense in $P(T)$ and $\delta = \omega_1 \cap N$ and R is a finite rectangle and $n = n(R)$ and $\bar{x} \in T_\delta^n$. Then there is $q \leq p$ such that $q \in D \cap N$ and $\heartsuit(\text{ht}(p), \bar{x}, f_q, R)$.

Proof: The proof of [JSL, Lemma 54] carries over verbatim.

Lemma 7.10. Suppose that λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ containing $P(T)$. Let $\delta = \omega_1 \cap N$, and suppose $p \in P(T) \cap N$. Suppose also that $\bar{x} \in T_\delta^n$ and R is a finite rectangle and $n = n(R)$. Then there is $q \leq p$ such that $\heartsuit(\text{ht}(p), \bar{x}, f_q, R)$ and $\text{ht}(q) = \delta$ and for every open dense $D \in N$ there is some $r \in D \cap N$ such that $q \leq r$. In particular, $P(T)$ is proper and does not add reals.

Proof: The proof of [JSL, Lemma 55] carries over verbatim.

For the definition of (S^*, ω_2) -p.i.c., see [JSL, Definition 39].

Lemma 7.11. $P(T)$ has (S^*, ω_2) -p.i.c.

Proof: The proof of [JSL, Lemma 57] carries over verbatim. However, please note the error in [JSL, Lemma 43] concerning the preservation of (S^*, ω_2) -p.i.c. The corrected statement of that Lemma is as follows:

Lemma 7.12. Suppose $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle Q_\eta : \eta < \kappa \rangle$. Suppose that for every $\eta < \kappa$ we have that $\mathbf{1} \Vdash_{P_\eta}$ “ Q_η has (S, ω_2) -p.i.c.” Then if $\kappa < \omega_2$ we have that P_κ has (S, ω_2) -p.i.c., and if $\kappa \leq \omega_2$ then P_κ has ω_2 -c.c.

Proof: [JSL, Lemma 43] neglects the restriction on the length of the iteration, and the proof given there is incorrect. The needed correction to the proof is to be found in the proof of [APAL, Lemma XXXX].

Lemma 7.13. $\mathbf{1} \Vdash_{P(T)}$ “ T is S^* -*-special.”

Proof: The proof of [JSL, Lemma 58] carries over verbatim.

The following is [PIF, remark on page XXX], [JSL, Lemma 8].

Lemma 7.14. Suppose $\langle P_\eta : \eta \leq \kappa \rangle$ is as in Lemma XXX, and suppose $\mathbf{1} \Vdash_{P_1}$ “ T is S^* -*-special.” Then $\mathbf{1} \Vdash_{P_\kappa}$ “ T is Aronszajn.”

Proof: See [JSL, Lemma 8].

We now turn to the task of showing that $P(T)$ is (T^*, S^*) -#-preserving, and hence by Lemmas 4.3, 4.4, and 4.5, and Theorem 5.4 we may construct the iteration so that in $V[G_{P_{\omega_2}}]$ we have:

- (1) T^* is Aronszajn,
- (2) for every Aronszajn tree T we have T is S^* -*-special,
- (3) for all $S' \subseteq \omega_1$ such that $S' - S^*$ is stationary, we have that T^* is not S' -*-special,
- (4) for all unbounded $E \subseteq \omega_1$ we have that T^* is not E -special.

Simply take T^* to be sharply Suslin in the ground model, and choose Q_0 to be $P(T^*)$ so that the hypotheses of Lemmas 4.5 and 7.14 are satisfied. In order to ensure (3), use posets of the form $CU(S^* \cup (\omega_1 - S_\eta))$ or $CU^*(S^* \cup (\omega_1 - S_\eta))$ or $CU^{**}(S^* \cup (\omega_1 - S_\eta))$, where $S_\eta = \{\text{rk}(x) : x \in \mathcal{I}_\eta\}$ where \mathcal{I}_η is (a name for) an antichain of T^* .

Lemma 7.15 (analogue of [JSL, Lemma 59]). *Suppose $P = P(T)$ and λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ containing $\{P, S^*, T^*\}$. Let $\delta = \omega_1 \cap N$. Suppose $m \in \omega$ and $\bar{x} \in T_\delta^m$ and $p \in P \cap N$ and $R \in N$ is a finite rectangle and $\bar{z} \leq \bar{x}$ and $\text{rk}(\bar{z}) = \text{ht}(p)$ and $A \in N$ is a P -name for a subset of T^* and x is (T^*, N) -#-generic. Then there is $y < x$ and $q \leq p$ such that $q \in N$ and $\heartsuit(\text{ht}(p), \bar{x}, f_q, R)$ and either $q \Vdash "y \in A"$ or $q \Vdash "(\forall z \geq y)(z \notin A)".$*

Proof: Let $A^* = \{y \in T^* : \text{for every } T\text{-promise } \Gamma \text{ such that } \text{rk}(\min(\Gamma)) \geq \max(\text{ht}(p), \text{rk}(y)) \text{ we have } \langle f_p, S_p, \Psi_p \cup \{\Gamma\} \rangle \Vdash "y \notin A"\}$. Notice $A^* \in N$, and therefore we may fix $y < x$ such that either $y \in A^*$ or $(\forall z \geq y)(z \notin A^*)$.

Case 1. $y \in A^*$.

Suppose there is no $q \leq p$ such that $q \Vdash "y \in A"$ and $\text{ht}(q) < \delta$ and $\heartsuit(\text{ht}(p), \bar{x}, f_q, R)$. Fix $\alpha \in C(\Psi_p)$ such that $\alpha \geq \max(\text{ht}(p), \text{rk}(y))$. Let $\Delta = \{\bar{w} \in T^n : \text{there is no } q \leq p \text{ such that } q \Vdash "y \in A" \text{ and } \text{ht}(q) < \text{rk}(\bar{w}) \text{ and } \heartsuit(\text{ht}(p), \bar{w}, f_q, R) \text{ and } \bar{w} \text{ is comparable with } \bar{x} \upharpoonright \alpha\}$. Notice $\Delta \in N$. We have that every $\bar{w} \leq \bar{x}$ is in Δ . Hence

$$N \models "\Delta \text{ is uncountable}."$$

We also have that Δ is downwards closed. Hence by Fact 7.5 we may take $\Gamma \subseteq \Delta$ such that Γ is a T -promise and $\min(\Gamma) \leq \bar{x}$ and $\text{rk}(\min(\Gamma)) \geq \max(\text{ht}(p), \text{rk}(y))$.

Because $y \in A^*$ we have that $\langle f_p, S_p, \Psi_p \cup \{\Gamma\} \rangle \nVdash "y \notin A."$ Therefore we may take $r \leq \langle f_p, S_p, \Psi_p \cup \{\Gamma\} \rangle$ such that $r \Vdash "y \in A."$ Because $\langle f_r, S_r \rangle$ fulfills Γ , we may take $\bar{w} \in \Gamma$ such that $\text{rk}(\bar{w}) > \text{ht}(r)$ and $\heartsuit(\text{ht}(p), \bar{w}, f_r, R)$. Because $\bar{w} \in \Delta$ there is no $q \leq p$ such that $q \Vdash "y \in A"$ and $\text{ht}(q) < \text{rk}(\bar{w})$ and $\heartsuit(\text{ht}(p), \bar{w}, f_q, R)$.

But r witnesses the opposite. This contradiction shows that if Case 1 holds then there is $q \leq p$ such that $q \Vdash "y \in A"$ and $\text{ht}(q) < \delta$ and $\heartsuit(\text{ht}(p), \bar{x}, f_q, R)$. Let $\beta = \text{ht}(q)$. We have $(\exists q \leq p)(\text{ht}(q) = \beta \text{ and } \heartsuit(\alpha, \bar{x} \upharpoonright \beta, f_q, R) \text{ and } q \Vdash "y \in A")$. Because $\bar{x} \upharpoonright \beta \in N$ we have $(\exists q \leq p)(\text{ht}(q) = \beta \text{ and } q \in N \text{ and } \heartsuit(\alpha, \bar{x} \upharpoonright \beta, f_q, R) \text{ and } q \Vdash "y \in A")$. Hence the conclusion of the Lemma holds in Case 1.

Case 2: Otherwise.

We have that $(\forall z \geq y)(z \notin A^*)$. Hence $(\forall z \geq y)(\exists \Gamma(z))(\Gamma(z) \text{ is a } T\text{-promise and } \text{rk}(\min(\Gamma)) \geq \max(\text{rk}(z), \text{ht}(p)) \text{ and } \langle f_p, S_p, \Psi_p \cup \{\Gamma(z)\} \rangle \Vdash "z \notin A")$. We may assume that the function mapping z to $\Gamma(z)$ is an element of N . Let $q = \langle f_p, S_p, \Psi_p \cup \{\Gamma(z) : z \geq y\} \rangle$. We have $q \in P(T) \cap N$ and $q \Vdash "(\forall z \geq y)(z \notin A)"$. Hence the conclusion of the Lemma holds in Case 2.

The Lemma is established.

Theorem 7.16. $P(T)$ is (T^*, S^*) -#-preserving.

Proof: Suppose λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ containing $\{P(T), T^*, S^*\}$. Let $\delta = \omega_1 \cap N$. If $\delta \in S^*$ then we are done by Lemma 7.10, so assume otherwise. Similarly, if there are no $x \in T_\delta^*$ such that x is (N, T^*) -#-generic, we are done, so assume otherwise. Suppose $p \in P(T) \cap N$. We build $q \leq p$ such that q is $(N, P(T), S^*, T^*)$ -#-preserving. Let $\langle D_m : m \in \omega \rangle$ list the dense open subsets of $P(T)$ that are in N . Let $\langle (\Gamma_m, R'_m, \bar{z}_m) : m \in \omega \rangle$ list all triples $\langle \Gamma, R', \bar{z} \rangle$ such that $\Gamma \in N$ is a T -promise and $R' \in N$ is a finite rectangle and $\bar{z} \in \Gamma \cap N$ and $n(R') = n(\Gamma)$, with infinitely many repetitions. Let $\langle (x_m, A_m) : m \in \omega \rangle$ list all pairs (x, A) such that x is (T^*, N) -#-generic and A is a $P(T)$ -name in N for a subset of T^* .

Build by recursion $\langle (F_m, q_m, p_m, \bar{w}_m) : m \in \omega \rangle$ such that $F_0 = \emptyset$ and $p_0 = p$ and each of the following holds:

- (1) F_m maps a finite subset of T_δ into the set of finite subsets of δ ,
- (2) $q_m \in P(T) \cap N$ and $q_m \leq p_m$ and $(\forall w \in \text{dom}(F_m))(\heartsuit(\text{ht}(p_m), w, f_{q_m}, F_m(w)))$ and for some $y < x_m$ we have either $q_m \Vdash "y \in A_m"$ or $q_m \Vdash "(\forall z \geq y)(z \notin A_m)"$,
- (3) $p_{m+1} \in D_m \cap N$ and $p_{m+1} \leq q_m$ and $(\forall w \in \text{dom}(F_m))(\heartsuit(\text{ht}(q_m), w, f_{p_{m+1}}, F_m(w)))$,
- (4) if $\Gamma_m \in \Psi_{p_{m+1}}$ and $\text{range}(\bar{z}_m) \subseteq \text{dom}(f_{p_{m+1}})$ then $\bar{w}_m \in \Gamma_m \cap T_\delta^{n(\Gamma_m)}$ and $\bar{z}_m \leq \bar{w}_m$ and for all $i < n(\Gamma_m)$ we have $\heartsuit(\text{rk}(\bar{z}_m), \bar{w}_m(i), f_{p_{m+1}}, (R'_m)_i)$ and $\text{range}(\bar{w}_m)$ is disjoint from $\text{dom}(F_m)$; otherwise, $\bar{w}_m = \emptyset$,
- (5) $\text{dom}(F_{m+1}) = \text{dom}(F_m) \cup \text{range}(\bar{w}_m)$,
- (6) for every $w \in \text{dom}(F_m)$ we have $F_{m+1}(w) \supseteq F_m(w)$,

(7) for all $i \in \text{dom}(\overline{w}_m)$ we have $F_{m+1}(\overline{w}_m(i)) \supseteq (R'_m)_i$.

The construction can be carried out using Lemma 7.15 to choose q_m as in (2) and Lemma 7.9 to choose p_{m+1} as in (3).

Let $q = \langle \bigcup \{f_{p_m} : m \in \omega\}, \bigcup \{S_{p_m} : m \in \omega\}, \bigcup \{\Psi_{p_m} : m \in \omega\} \rangle$. It is easy to see that q is as required.

The theorem is established.

We turn to the problem of showing that the forcing iteration under consideration does not add reals.

For the definition of “ $(S, < \omega_1)$ -proper,” see [JSL, Definition 26].

Lemma 7.17. *$P(T)$ is $(\omega_1, < \omega_1)$ -proper, and hence $P(T)$ is $(S, < \omega_1)$ -proper for any stationary $S \subseteq \omega_1$.*

The proof of [JSL, Lemma 56] carries over verbatim, but there is a small error: namely, in Case 1 it is implicitly assumed that $\gamma \neq 0$. Fortunately, the case that $\gamma = 0$ is easily handled by simply setting $r^* = p$.

The following is [JSL, Definition 27], based on [PIF, Chapter V].

Definition 7.18. *Suppose $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration. We say that the iteration is strictly $(S, < \omega_1)$ -proper iff **whenever** $\rho < \omega_1$ and λ is a sufficiently large regular cardinal and $\langle N_i : i \leq \rho \rangle$ is a continuous tower of countable elementary substructures of H_λ and $P_\kappa \in N_0$ and for every $i < \rho$ we have $\langle N_j : j \leq i \rangle \in N_{i+1}$ and for every $i \leq \rho$ we have $\omega_1 \cap N_i \in S$ and $i \in N_i$ and $\eta \in \kappa \cap N_0$ and $p \in P_\eta$ and for every $i \leq \rho$ we have that p is (N_i, P_η) -generic, and $p \Vdash$ “ $q \in P_{\eta, \kappa} \cap N_0[G_{P_\eta}]$,” **then** there is $r \in P_\kappa$ such that $r \restriction \eta = p$ and $p \Vdash$ “ $r \restriction [\eta, \kappa] \leq q$ ” and for every $i \leq \rho$ we have that r is (N_i, P_κ) -generic and $\text{supt}(r) \subseteq \eta \cup N_\rho$.*

Definition 7.19. *Suppose λ is large for P and M is a countable elementary substructure of H_λ and $P \in M$ and $p \in P \cap M$. We set $\text{Gen}(M, P, p)$ equal to the set of all $G \subseteq P \cap M$ which satisfy all of the following:*

- (1) G is M -generic, i.e., whenever $D \in M$ is a dense open subset of P then $G \cap D \neq \emptyset$
- (2) G is directed, i.e., $(\forall q_1 \in G)(\forall q_2 \in G)(\exists r \in G)(r \leq q_1 \text{ and } r \leq q_2)$
- (3) $p \in G$

Definition 7.20. *Suppose that S is stationary and P is S -proper not adding reals and suppose Q is a P -name for a poset. We say Q is S -complete for P iff*

whenever λ is a sufficiently large regular cardinal and M and N are countable elementary substructures of H_λ and $P * Q \in M \in N$ and $\omega_1 \cap M \in S$ and $\omega_1 \cap N \in S$ and $q \in M$ is a P -name for an element of Q and $G \in \text{Gen}(M, P, \mathbf{1}) \cap N$, then there is $G' \in \text{Gen}(M, P * Q, \mathbf{1})$ such that $\{p_1 \in P : (\exists r)((p_1, r) \in G')\} = G$ and $(\mathbf{1}, q) \in G'$ and whenever p is a lower bound for G and p is N -generic, then there is $p' \leq p$ and a P -name s for an element of Q such that (p', s) is a lower bound for G' .

Lemma 7.21. *Suppose $S^* \subseteq S$. Then $CU(S)$ and $CU^*(S)$ are $(S^*, < \omega_1)$ -proper. Furthermore, for every P such that P is S^* -proper not adding reals, if $\mathbf{1} \Vdash_P "Q = CU(S) \text{ or } Q = CU^*(S) \text{ and } S^* \subseteq S"$ then Q is S^* -complete for P .*

Proof: See [JSL, Lemmas 37 and 38].

Lemma 7.22. *Suppose $S \subseteq \omega_1$ is stationary and $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support iteration based on $\langle Q_\eta : \eta < \kappa \rangle$ and for every $\eta < \kappa$ we have that Q_η is S -complete for P_η , and suppose also that P_κ is strictly $(S, < \omega_1)$ -proper. Then P_κ does not add reals.*

Proof: See [JSL, Theorem 36].

Theorem 7.23. *If ZFC is consistent, then so is ZFC plus there is a stationary co-stationary set S^* such that every Aronszajn tree is S^* -*-special plus there is an Aronszajn tree T^* such that T^* is not S^* -*-special whenever $S - S^*$ is stationary, and for every unbounded $E \subseteq \omega_1$ we have that T^* is not E -special. Furthermore, we may either have CH hold or CH fail in the model.*

Proof: This is [JSL, Theorem 45] with three changes. The first change is that we allow forcings of the form $CU^*(S^* \cup (\omega_1 - S_\eta))$ and $CU^{**}(S^* \cup (\omega_1 - S_\eta))$. Naturally, by using $CU^{**}(S^* \cup (\omega_1 - S_\eta))$ we will not have CH in the final model. The second change is that we start with a sharply Suslin tree and have the property of (S^*, T^*) -#-preserving in order to assure that for every unbounded $E \subseteq \omega_1$ we have $\mathbf{1} \Vdash_{P_{\omega_2}} "T^* \text{ is not } E\text{-special}."$ The third change is that we use the version of $P(T)$ in which Ψ is not required to be countable, but only that for each α we have that $\{\Gamma \in \Psi : \text{rk}(\min(\Gamma)) \leq \alpha\}$ is countable.

8 The “no stationary antichains” models

In [APAL] a model of ZFC plus SH plus some Aronszajn tree has no stationary antichain is constructed. The Suslin trees are killed more gently than in [PIF, Chapter IX.4] and [PIF] and [STACY], because there is an Aronszajn tree T^* such that for every stationary $S \subseteq \omega_1$ we have that T^* is not S -*-special. Here we show how to kill Suslin trees even more gently; in the final model, Suslin’s hypothesis holds and there is an Aronszajn tree T^* such that for every stationary S and every unbounded E we have that T is neither S -*-special nor E -special.

First we recall the differences between the construction of [JSL] and [APAL].

The first difference is the poset that is used. Before exhibiting the poset from [APAL], we give some definitions.

Definition 8.1. *Suppose T is an Aronszajn tree and f is a monotonically non-decreasing function from $\bigcup\{T_\beta : \beta \leq \alpha\}$ into $\{0, 1\}$. We set $\text{ht}(f)$ equal to α . Given $\bar{z} \in T^n$ and $\rho < \omega_1$, we say $\heartsuit(\rho, f, \bar{z})$ iff either $\rho \geq \text{rk}(\bar{z})$ or for all $i < n$ and all $t \leq \bar{z}(i)$ such that $t \in \text{dom}(f)$ we have $f(t) = f(\bar{z}(i) \restriction \rho)$. Given a T -promise Γ , we say that f fulfills Γ iff whenever $\beta < \gamma$ are in $C(\Gamma)$ and $\beta < \text{ht}(f)$ and $\bar{w} \in \Gamma \cap T_\beta^{n(\Gamma)}$, then there is an infinite $W \subseteq \Gamma \cap T_\gamma^n$ such that distinct elements of W have disjoint ranges and for every $\bar{w} \in W$ we have $\heartsuit(\beta, f, \bar{w})$.*

In [APAL] we may view the poset $P(T)$ as $\{\langle f, \Psi \rangle : \text{for some } \alpha < \omega_1 \text{ we have that } f \text{ is a monotonically non-decreasing function from } \bigcup\{T_\beta : \beta \leq \alpha\} \text{ into } \{0, 1\} \text{ and } \Psi \text{ is a countable set of promises that } f \text{ fulfills}\}$.

We have $\mathbf{1} \Vdash_{P(T)} “T \text{ is not Suslin because } \{t \in T : t \text{ has an immediate predecessor } t' \text{ such that } f(t') = 0 \text{ and } f(t) = 1, \text{ where } f = \bigcup\{f' : (\exists \Psi)(\langle f, \Psi \rangle \in G_{P(T)})\}.”$

The second change is that we must ensure that the tree T^* must remain Aronszajn in $V[G_{P_{\omega_2}}]$. Recall that in [JSL] (and in [PIF, Chapter IX.4]) this was accomplished by S^* -*-specializing T^* at the first step of the iteration, so that it could not become non-Aronszajn in any extension in which ω_1 is not collapsed. This strategy is not available in the construction in [APAL]. Instead, the fact that T^* remains Aronszajn is ensured by showing that the iteration satisfies a preservation property that is more stringent than the property (T^*, S^*) -preserving used in [PIF, Section IX.4] and [JSL]. See [APAL, Definition XXX] for the definition of this property, [APAL, Lemma XXX] for the fact that the property ensures that T^* remains Aronszajn, and various Lemmas in [APAL] for the fact that the property is preserved under the appropriate iterations and that the property is

satisfied by the constituent posets of the iteration.

These two changes were enough to carry out the construction of [APAL], but left the question of whether we could arrange for CH to hold in the final model. The difficulty was resolved in [APAL2] by using the following strategy.

9 Doing without Kurepa trees

In this section, we strengthen the conclusion of Theorem 7.23 by requiring that every ω_1 -tree is S^* -*-special. Therefore we have Kurepa's hypothesis holds in the model. Naturally, this requires that the hypothesis be strengthened from the consistency of ZFC to the consistency of ZFC plus there exists an inaccessible cardinal. We use (essentially) the forcing from [STACY].

We repeat the main Definitions from [STACY]. We fix T to be an ω_1 -tree and T^* an Aronszajn tree and B equal to the set of uncountable branches of T and S^* a stationary co-stationary subset of ω_1 . Fix κ a sufficiently large regular cardinal. For $n \in \omega$ we set T^n equal to $\{\bar{w} : \bar{w} \text{ is a function with domain } n \text{ and there is some } \alpha < \omega_1 \text{ such that } (\forall i < n)(\bar{w}(i) \in T_\alpha)\}$. Notice that for every n we have that T^n is an ω_1 -tree.

Definition 9.1. Γ is a promise iff there is $n = n(\Gamma) \in \omega$ and $C = C(\Gamma) \subseteq \omega_1$ closed unbounded and $\bar{x} = \min(\Gamma) \in T^{n(\Gamma)}$ and $G = G(\Gamma) \subseteq n$ and $\langle b_i(\Gamma) : i \in G \rangle$ a sequence of elements of B such that $\bar{x} \in \Gamma \subseteq T_C^n$ and $(\forall \bar{y} \in \Gamma)(\bar{x} \leq \bar{y})$ and for all $\alpha < \beta$ both in C and every $\bar{y} \in \Gamma \cap T_\alpha^{n(\Gamma)}$ then there is $W \subseteq \Gamma \cap T_\beta^{n(\Gamma)}$ such that $(\forall \bar{w} \in W)(\bar{y} < \bar{w} \text{ and } (\forall \bar{w}' \in W)(\text{either } \bar{w}' = \bar{w} \text{ or } \{\bar{w}'(i) : i \in n(\Gamma) - G\} \text{ is disjoint from } \{\bar{w}(i) : i \in n(\Gamma) - G\}))$, and for all $\bar{y} \in \Gamma$ and $i \in G$ we have $\bar{y}(i) \in b_i(\Gamma)$, and W is infinite unless $G = n$.

Notwithsatndng the fact that we have redefined the notion of “promise,” we keep the same definition of “finite rectangle” (Definition 7.1).

Definition 9.2. Suppose $n \in \omega$ and $\bar{w} \in T^n$ and R is a finite rectangle and $n(R) = n$ and f is a function from a subset of T into ω_1 . Then we say $\heartsuit(\alpha, \bar{w}, f, R)$ iff $(\forall i < n)(\forall y \leq \bar{w}(i))(\text{if } \text{rk}(y) > \alpha \text{ and } y \in \text{dom}(f) \text{ then } f(y) \notin R(i))$. For b an uncountable branch of T we say $\heartsuit(\alpha, b, f, R)$ iff $n(R) = 1$ and $(\forall x \in b)(\heartsuit(\alpha, x, f, R))$.

Definition 9.3. Suppose S is a bounded subset of ω_1 and f is a function that S -*-specializes T and $n \in \omega$ and Γ is a promise and $n = n(\Gamma)$. We say that

$\langle f, S \rangle$ fulfills Γ iff $S - \text{rk}(\min(\Gamma)) \subseteq C(\Gamma)$ and for every $\alpha < \beta$ both in $C(\Gamma)$ and every finite rectangle R with $\text{dom}(R) = n$ and every $\bar{y} \in \Gamma \cap T_\alpha^n$ there is $W \subseteq \Gamma \cap T_\beta^n$ such that either $G(\Gamma) = n$ or W is infinite, and such that for every \bar{w} and \bar{w}' distinct elements of W we have that $\{\bar{w}(i) : i \in n - G(\Gamma)\} \cap \{\bar{w}'(i) : i \in n - G(\Gamma)\} = \emptyset$ and, for every $\bar{w} \in W$, we have $\bar{y} \leq \bar{w}$ and $\heartsuit(\alpha, \bar{w}, f, R)$.

Definition 9.4. We set $P = P(T, \kappa)$ equal to the set of all $\langle f, S, \mathcal{N}, \Psi \rangle$ such that

- (1) S is a countable set of countable limit ordinals,
- (2) f is a function that S -*-specializes T ,
- (3) for some non-limit $\alpha =_{\text{def}} \text{lh}(\mathcal{N}) < \omega_1$ we have that $\mathcal{N} = \langle \mathcal{N}(i) : i < \alpha \rangle$ is a tower (not necessarily continuous) of countable elementary substructures of H_κ ,
- (4) for $i < j < \text{lh}(\mathcal{N})$ we have $\mathcal{N}(i) \in \mathcal{N}(j)$ and if $\alpha \neq 0$ then $\{T, T^*, S^*, B\} \in \mathcal{N}(0)$, and for every $i < \text{lh}(\mathcal{N})$ we have $\omega_1 \cap \mathcal{N}(i) \in S$,
- (5) Ψ is a set of promises that $\langle f, S \rangle$ fulfills,
- (6) for every $\beta < \omega_1$ we have that $\{\Gamma \in \Psi : \text{rk}(\min(\Gamma)) < \beta\}$ is countable,
- (7) for every limit ordinal α and $\gamma \in S^*$, if $\{\omega_1 \cap \mathcal{N}(\beta) : \beta < \alpha\}$ is unbounded in γ then $\alpha \in \text{dom}(\mathcal{N})$ and $\mathcal{N}(\alpha) = \bigcup \{\mathcal{N}(\beta) : \beta < \alpha\}$,
- (8) for all $\beta \in \text{dom}(\mathcal{N})$, for all $x \in \text{dom}(f) - \mathcal{N}(\beta)$ the following are equivalent:
 - i) $(\exists y < x)(y \in \text{dom}(f) \cap \mathcal{N}(\beta) \text{ and } f(x) = f(y))$
 - ii) $(\exists b \in B \cap \mathcal{N}(\beta))(x \in b)$

We order P by declaring $\langle f, S, \mathcal{N}, \Psi \rangle \leq \langle f', S', \mathcal{N}', \Psi' \rangle$ iff S end-extends S' and $f' \subseteq f$ and \mathcal{N} end extends \mathcal{N}' and $\Psi' \subseteq \Psi$.

Notations 9.5. For $p \in P$ we give f_p, S_p, \mathcal{N}_p , and Ψ_p their obvious meanings, and we set $\text{ht}(p) = \sup(S_p)$ and $L_p = \bigcup \{\mathcal{N}_p(i) : i \in \text{dom}(\mathcal{N}_p)\}$ and for $x \in L_p$ we let $\rho_p(x)$ denote the least γ such that $x \in \mathcal{N}_p(\gamma)$. For $b \in B \cap L_p$ we set $\delta_p(b) = \omega_1 \cap \mathcal{N}_p(\rho_p(b))$ and we let $\mu_p(b)$ denote the unique $x \in b$ such that $\text{rk}(x) = \delta_p(b)$, and if $\delta_p(b) \in S_p$ then we set $\sigma_p(b)$ equal to $f_p(\mu_p(b))$. We set $U_p = \{x \in T : (\exists b \in B \cap L_p)(x \in b)\}$.

Lemma 9.6. Suppose λ is a sufficiently large regular cardinal (in particular, much larger than κ) and M is a countable elementary substructure of H_λ containing $\{P, T, T^*, S^*, B, \kappa\}$. Suppose $p \in M \cap P$ and $n \in \omega$. Let $\delta = \omega_1 \cap M$. Suppose $\bar{x} \in T_\delta^n$ and suppose R is a finite rectangle and $n(R) = n$. Suppose

$z \in M \cap H_\kappa$. Then there is $q \in P \cap M$ such that $q \leq p$ and $z \in L_q$ and $\heartsuit(\text{ht}(p), \bar{x}, f_q, R)$.

Proof: Let N be a countable elementary substructure of H_κ such that $N \in M$ and $\{z, R \cap \delta^n, T, T^*, S^*\} \in N$, and let $\delta_\omega = \omega_1 \cap N$. Choose $\langle \delta_m : m < \omega \rangle$ an increasing sequence from $\delta_\omega \cap N$ cofinal in δ_ω such that $\omega_1 \cap L_p < \delta_0$ and $(\forall m \in \omega) (\delta_m \in C(\Psi_p))$, where $C(\Psi_p)$ is as in the paragraph following Definition 7.7, and such that for every $m \in \omega$, for every $\bar{y} \in \Gamma \cap T_{\delta_m}^{n(\Gamma)}$ there is $W \subseteq \Gamma \cap T_{\delta_{m+1}}^{n(\Gamma)}$ such that for every $\{\bar{w}, \bar{w}'\} \subseteq W$, if $\bar{w} \neq \bar{w}'$ then $\{\bar{w}(i) : i \in n(\Gamma) - G(\Gamma)\}$ is disjoint from $\{\bar{w}'(i) : i \in n(\Gamma) - G(\Gamma)\}$ and either W is infinite or $G(\Gamma) = n(\Gamma)$.

Let $\langle (\bar{y}_k, \Gamma_k, R'_k, t_k) : k \in \omega \rangle \in M$ list all quadruples (\bar{y}, Γ, R', t) such that $\Gamma \in \Psi_p$ and $\bar{y} \in \Gamma \cap N$ and $R' \subseteq \delta_\omega^{n(\Gamma)}$ is a finite rectangle and $t \leq \omega$ and $(\forall i \in n(\Gamma))(i \in G(\Gamma) \text{ iff } (\exists b \in B \cap L_p)(\min(\Gamma)(i) \in b))$, with each such quadruple listed infinitely many times.

Let $\langle x_i : i \in \omega \rangle \in M$ list $\bigcup \{T_{\delta_m} : m \leq \omega\}$.

Working in M , build $\langle (\bar{z}_m, Z_m, x_m^\#, X_m, f_m) : m \in \omega \rangle$ such that $f_0 = f_p$ and for every $m \in \omega$ each of the following holds:

- (1) $f_m \subseteq f_{m+1}$
- (2) $\text{dom}(f_{m+1}) = \text{dom}(f_m) \cup \{x_m\} \cup Z_m \cup X_m$
- (3) if $\delta_{t_m} > \text{rk}(\bar{y}_m)$ then $\bar{y}_m < \bar{z}_m$ and $\text{rk}(\bar{z}_m) = \delta_{t_m}$ and $\{\bar{z}_m(i) : i \in G(\Gamma_m)\} = U_p \cap \text{range}(\bar{z}_m)$ and $\bar{z}_m \in \Gamma_m$
- (4) $Z_m = \{\bar{z}_m(i) : i \in n(\Gamma_m)\}$
- (5) if $\delta_{t_m} \leq \text{rk}(\bar{y}_m)$ then $Z_m = \emptyset$
- (6) if $\text{rk}(x_m) \neq \delta_\omega$ or $x_m \in \text{dom}(f_m) \cup Z_m$ or there is no $b \in B \cap N$ such that $x_m \in b$ then $X_m = \emptyset$; otherwise, $j_m \in \omega$ is large enough that $(\forall x \in \text{dom}(f_m) \cup Z_m)(x_m \upharpoonright \delta_{j_m} \not\leq x)$ and $x_m^\# < x_m$ and $\text{rk}(x_m^\#) = \delta_{j_m}$ and $X_m = \{x_m^\#\}$
- (7) if $X_m \neq \emptyset$ then $f_{m+1}(x_m^\#) = f_{m+1}(x_m)$
- (8) for all $x \in \text{dom}(f_{m+1})$, if there is $b \in B \cap L_p$ such that $x \in b$ then for the unique such b we have $f_{m+1}(x) = \sigma_p(b)$
- (9) for every $x \in \text{dom}(f_{m+1} - f_m)$, if there is no $b \in B \cap L_p$ such that $x \in b$ then $f_{m+1}(x) \notin \{f_{m+1}(x') : x' \in \text{dom}(f_m) \cup \{x_m\} \cup Z_m\}$
- (10) for all $j \leq m$, if $\delta_{t_j} > \text{rk}(\bar{y}_j)$ then $\heartsuit(\text{rk}(\bar{y}_j), \bar{z}_j, f_{m+1}, R'_j)$
- (11) $\heartsuit(\text{ht}(p), \bar{x}, f_m, R)$

There is no difficulty in meeting these requirements. Set $q = \langle \bigcup \{f_m : m \in \omega\}, S_p \cup \{\delta_m : m \leq \omega\}, \mathcal{N}_p \hat{\ } \langle N \rangle, \Psi_p \rangle$. Then q is as required in the conclusion of the Lemma.

Lemma 9.7. *Suppose T is an ω_1 -tree and $k \in \omega$ and $\Delta \subseteq T^k$ is uncountable and downward closed, and suppose $\bar{x} \in T^k$ and every element of Δ is comparable with \bar{x} . Then there is a promise $\Gamma \subseteq \Delta$ such that $\min(\Gamma) = \bar{x}$.*

Proof: We build $G \subseteq k$ and a sequence of uncountable branches $\langle b_i : i \in G \rangle$ in stages as follows.

Initially, let $G_0 = \emptyset$ and $\Delta_0 = \Delta$ and $T^0 = \{x \in T : (\exists \bar{y} \in \Delta)(\exists i < k)(x = \bar{y}(i))\}$.

Stage j :

Case 1: T^j is Aronszajn.

Take $G = G_j$. By Fact 7.5 we may take $\Gamma' \subseteq \Delta_j$ such that for every $i \in k - G$ we have $\min(\Gamma')(i) = \bar{x}(i)$. Let $\Gamma = \{\bar{y} \in \Delta : (\exists \bar{z} \in \Gamma')((\forall m \in k - G)(\bar{y}(m) = \bar{z}(m)) \text{ and } (\forall m \in G)(\bar{y}(m) \text{ is the unique element of } b_m \cap T_{\text{rk}(\bar{y})}))\}$.

Case 2: Otherwise.

Let b be an uncountable branch of T^j and fix $i \in k - G_j$ such that $b \cap \{\bar{y}(i) : \bar{y} \in \Delta_j\}$ is uncountable. Denote this b by b_i . Notice that because Δ_j is downwards closed, we have that b_i is a subset of $\{\bar{y}(i) : \bar{y} \in \Delta_j\}$. Let $G_{j+1} = G_j \cup \{i\}$. For every $\bar{y} \in \Delta_j$ let $s(\bar{y}) \in T^{k-G_{j+1}}$ be defined by $\text{dom}(s(\bar{y})) = k - G_{j+1}$ and for every $m \in k - G_{j+1}$ we have $s(\bar{y})(m) = \bar{y}(m)$. Set $\Delta_{j+1} = \{s(\bar{y}) : \bar{y} \in \Delta_j\}$, and let $T^{j+1} = \{\bar{y}(m) : \bar{y} \in \Delta_{j+1} \text{ and } m \in k - G_{j+1}\}$. Now proceed to Stage $j + 1$.

The Lemma is established.

Lemma 9.8. *Suppose λ is a sufficiently large regular cardinal and M is a countable elementary substructure of H_λ such that $\{P, T, T^*, S^*, B, \kappa\} \in M$. Suppose $n \in \omega$ and $\bar{x} \in T_{\omega_1 \cap M}^n$ and R is a finite rectangle with $n(R) = n$. Suppose $p \in P \cap M$. Then whenever $D \in M$ is a dense open subset of P , there is $q \leq p$ such that $q \in D \cap M$ and $\heartsuit(\text{ht}(p), \bar{x}, f_q, R)$.*

Proof: Suppose D is a counterexample. We may assume $R \in M$ because if we replace each $R(i)$ with $R(i) \cap M$ we do not thereby change the truth of $\heartsuit(\text{ht}(p), \bar{x}, f, R)$ for any $f \in M$. Set $\bar{z} = \bar{x} \restriction \text{ht}(p)$ and set $\Delta = \{\bar{y} \in T^n : \bar{y} \text{ is comparable with } \bar{z} \text{ and there is no } q \leq p \text{ such that } q \in D \text{ and } \text{ht}(q) \leq \text{rk}(\bar{y}) \text{ and } \heartsuit(\text{ht}(p), \bar{y}, f_q, R)\}$. Notice that Δ is downward closed and $\{\bar{y} \in T^n : \bar{y} < \bar{x}\} \subseteq \Delta$. Necessarily Δ is uncountable because $M \models “(\forall \alpha < \omega_1)(\Delta \cap T_\alpha^n \neq \emptyset)”$. Therefore by Lemma 9.7 we may take $\Gamma \subseteq \Delta$ a promise with $\min(\Gamma) = \bar{z}$.

Let $p' = \langle f_p, S_p, \mathcal{N}_p, \Psi_p \cup \{\Gamma\} \rangle$. Take $r \leq p'$ such that $r \in D$. Because $\langle f_r, S_r, \mathcal{N}_r \rangle$ fulfills Γ , we may take $\bar{w} \in \Gamma$ with $\text{ht}(r) = \text{rk}(\bar{w})$ and $\heartsuit(\text{ht}(p), \bar{w}, f_r, R)$.

Because $\overline{w} \in \Delta$, there is no $q \leq p$ such that $q \in D$ and $\text{ht}(q) \leq \text{rk}(\overline{w})$ and $\heartsuit(\text{ht}(p), \overline{w}, f_q, R)$. But r is a witness that there is such a q . This contradiction establishes the Lemma.

Lemma 9.9. *Suppose λ is a sufficiently large regular cardinal and M is a countable elementary substructure of H_λ such that $\{P, T, T^*, S^*, B, \kappa\} \in M$. Suppose $n \in \omega$ and $\overline{x} \in T_{\omega_1 \cap M}^n$ and R is a finite rectangle with $n(R) = n$. Suppose $p \in P \cap M$. Then there is $q \leq p$ such that q is M -generic and $\text{ht}(q) = \omega_1 \cap M$ and $\heartsuit(\text{ht}(p), \overline{x}, f_q, R)$.*

Proof: Take N a countable elementary substructure of H_κ such that $N \in M$ and $\{p, \zeta, T, \overline{A} \cap \delta^n\} \in N$. Set $\delta_\omega = \omega_1 \cap N$ and choose $\langle \delta_m : m \in \omega \rangle$ an increasing sequence from $\delta_\omega \cap N$ cofinal in δ_ω , such that for every $\Gamma \in \Psi_p$ and every $m \in \omega$ we have $\delta_m \in C(\Gamma)$ and for every $\overline{y} \in \Gamma \cap T_{\delta_m}^{n(\Gamma)}$ there is $W \subseteq \Gamma \cap T_{\delta_{m+1}}^{n(\Gamma)}$ such that for every $\overline{w} \in W$ we have $\overline{y} \leq \overline{w}$ and for every $\{\overline{w}, \overline{w}'\} \subseteq W$, if $\overline{w} \neq \overline{w}'$ then $\{\overline{w}(i) : i \in n(\Gamma) - G(\Gamma)\}$ is disjoint from $\{\overline{w}'(i) : i \in n(\Gamma) - G(\Gamma)\}$, and either W is infinite or $G(\Gamma) = n(\Gamma)$.

Let $\langle \langle \overline{y}_k, \Gamma_k, \overline{A}_k^*, t_k \rangle : k \in \omega \rangle$ list all $\langle \overline{y}, \Gamma, \overline{A}^*, t \rangle$ such that $\Gamma \in \Psi_p$ and $\overline{y} \in \Gamma \cap N$ and $\overline{A}^* \subseteq (\delta_\omega)^n(\Gamma)$ is a finite rectangle, and $t \leq \omega$, listed with infinitely many repetitions.

Let $S_q = S_p \cup \{\delta_t : t \leq \omega\}$. Let $\langle x_m : m \in \omega \rangle$ list $\bigcup \{T_{\delta_m} : m \leq \omega\}$.

Build $\langle f_m : m \in \omega \rangle$ such that $f_0 = f_p$ and each of the following holds:

- (1) $f_m \subseteq f_{m+1}$ and $\text{dom}(f_{m+1}) = \text{dom}(f_m) \cup \{x_m\} \cup Z_m \cup X_m$
- (2) if $\delta_{t_m} > \text{rk}(\overline{y}_m)$ then $\overline{z}_m > \overline{y}_m$ and $\text{rk}(\overline{z}_m) = \delta_{t_m}$ and $\{\overline{z}_m(i) : i \in n(\Gamma_m) - G(\Gamma_m)\}$ is disjoint from $\text{dom}(f_m)$ and $\overline{z}_m \in \Gamma_m$ and $Z_m = \{\overline{z}_m(i) : i \in n(\Gamma_m)\}$
- (3) if $\delta_{t_m} \leq \text{rk}(\overline{y}_m)$ then $Z_m = \emptyset$
- (4) if $t_m \neq \omega$ or $G(\Gamma_m) = \emptyset$ then $X_m = \emptyset$; otherwise, $j_m \in \omega$ is large enough that $x_m \restriction \delta_{j_m} \not\leq x_j$ for all $j < m$, and $X_m = \{x_m^\#\}$ where $x_m^\# = \overline{w}_m^\#(i)$ for some $i \in G(\Gamma_m)$ and some $\overline{w}_m^\# \in \Gamma_m \cap T_{\delta_{j_m}}^{n(\Gamma_m)}$, and $\overline{w}_m^\#(i) < x_m$ and $(\forall b \in B \cap L_p) (\overline{w}_m^\#(i) \in b \text{ implies } x_m \in b)$
- (5) if $X_m \neq \emptyset$ then $f_{m+1}(x_m^\#) = f_{m+1}(x_m)$
- (6) if there is $b \in B \cap L_p$ such that $x_m \in b$ then for the unique such b we have $f_{m+1}(x_m) = \sigma_p(b)$
- (7) if $\delta_m > \text{rk}(\overline{y}_m)$ then for all $i < n(\Gamma_m)$, if for some $b \in B \cap L_p$ we have $\overline{z}_m(i) \in b$, then for the unique such b we have $f_{m+1}(\overline{z}_m(i)) = \sigma_p(b)$
- (8) for every $j \leq m$ such that $\delta_{t_j} > \text{rk}(\overline{y}_j)$ we have $\heartsuit(\overline{z}_j, f_m, \overline{a}_j^*)$ implies

$\heartsuit(\bar{z}_j, f_{m+1}, \bar{a}_j^*)$, and for all $i \in n(\Gamma_j) - G(\Gamma_j)$ we have $f_{m+1}(\bar{z}_j(i)) \neq f_m(z)$ for all $z \in \text{dom}(f_m)$ such that z is comparable with $\bar{z}_j(i)$

(9) if $x_m \notin \text{dom}(f_m) \cup Z_m \cup X_m \cup (\bigcup(B \cap L_p))$ then $f_{m+1}(x_m) \notin \{f_{m+1}(t) : t \in \text{dom}(f_m) \cup Z_m \cup X_m\}$

(10) $\heartsuit(\bar{x}, f_m, \bar{A})$ implies $\heartsuit(\bar{x}, f_{m+1}, \bar{A})$

Let $f_q = \bigcup\{f_m : m \in \omega\}$ and $\mathcal{N}_q = \mathcal{N}_p \hat{\setminus} N$.

10 Not adding reals

In this section we discuss a sufficient condition for no reals to be added. This condition has two parts. One part is a generalization of [18, Definition 32], which is a variant of Shelah's notion of \mathcal{D} -completeness [23, Chapter V]. The second part is Definition 32 given above.

Lemma 41. *Suppose \dot{Q} is (T, X) -complete for P and λ is large for $\{P * \dot{Q}, X\}$ and $M \prec N$ are countable elementary substructures of H_λ and $\{P * \dot{Q}, X, T\} \in M \in N$ and $(p, \dot{q}) \in P * \dot{Q} \cap M$ and $G \in \text{Gen}(M, T, P, p) \cap N$. Then there are a P -name \dot{s} and a set $G' \in \text{Gen}(M, T, P * \dot{Q}, (p, \dot{q}))$ such that $G = \{p' \in P : (\exists \dot{r})(\langle p', \dot{r} \rangle \in G')\}$ and whenever \tilde{p} is a lower bound for G which is (M, P, T) -completely preserving and (N, P, T) -preserving then (\tilde{p}, \dot{s}) is an (M, P, T) -completely preserving lower bound for G' .*

Proof: Let G' be as in the conclusion of Definition 39 and for every $p' \in P$ such that p' is a lower bound for G which is both (M, P, T) -completely preserving and (N, P, T) -preserving let $\dot{s}(p')$ be as in the conclusion of Definition 39. Let \mathcal{J} be a maximal antichain of the set of such p' and take \dot{s} such that $(\forall p' \in \mathcal{J}) (p' \Vdash \dot{s} = \dot{s}(p'))$. We have that G' and \dot{s} are as required.

Definition 42. *Suppose $\langle P_\eta : \eta \leq \alpha \rangle$ is a countable support iteration and T is Suslin and X is any set. We say that P_α is (T, X) -strictly complete iff **whenever** λ is large for $\{P_\alpha, X\}$ and M is a countable elementary substructure of H_λ and $\{P_\alpha, X, T\} \in M$ and α^* is the order-type of $\alpha \cap M$ and $\mathcal{N} = \langle N_i : i \leq \alpha^* \rangle$ is a λ -tower for M and $p \in P_\alpha \cap M$ and $\eta \in \alpha \cap M$ and η^* is the order-type of $\eta \cap M$ and $G \in \text{Gen}(M, P_\eta, p \restriction \eta) \cap N_{\eta^*+1}$ **then** there are $G' \in \text{Gen}(M, P_\alpha, p)$ and a P_η -name \dot{s} such that $\{r \restriction \eta : r \in G'\} = G$ and **whenever** \tilde{p} is an (M, P_η, T) -completely preserving lower bound for G and \tilde{p} is $(\langle N_i : \eta^* < i \leq \alpha^* \rangle, P_\eta, T)$ -preserving **then***

we have that there is $\tilde{s} \in P_\alpha$ such that $\tilde{s} \upharpoonright \eta = \tilde{p}$ and \tilde{s} is an (M, P_α, T) -completely preserving lower bound for G' and $\tilde{p} \Vdash \text{"}\tilde{s} \upharpoonright [\eta, \alpha) = \dot{s}\text{"}$ and $\text{supt}(\tilde{s}) \subseteq \eta \cup N_{\alpha^*}$.

Lemma 43. Suppose P_α is (T, X) -strictly complete for some X . Then P_α does not add reals.

Proof: Simply take $\eta = 0$ in Definition 42.

Lemma 44. Suppose $\langle P_\eta : \eta \leq \alpha \rangle$ is a countable support iteration based on $\langle \dot{Q}_\eta : \eta < \alpha \rangle$ and T is Suslin and for all $\eta < \alpha$ we have \dot{Q}_η is (T, X_η) -complete for P_η , and suppose for every $\beta \leq \alpha$ we have that P_β is strictly strongly T -preserving. Then P_α is (T, X) -strictly complete where $X = \langle X_\eta : \eta < \alpha \rangle$.

Proof: We work by induction on α . Let $\lambda, M, \alpha^*, \eta, \eta^*, \mathcal{N} = \langle N_i : i \leq \alpha^* \rangle, p$, and G be as in the hypothesis of Definition 42.

Suppose first that $\alpha = \beta + 1$. Let β^* be the order-type of $\beta \cap M$. By the induction hypothesis we may take $G_1 \in \text{Gen}(M, P_\beta, p \upharpoonright \beta)$ and \dot{s}_1 such that $\{r \upharpoonright \eta : r \in G_1\} = G$ and whenever \tilde{p} is a lower bound for G which is both (M, P_η, T) -completely preserving and $(\langle N_i : \eta^* < i \leq \beta^* \rangle, P_\eta, T)$ -preserving then we have that there is $s^* \in P_\beta$ such that s^* is an (M, P_β, T) -completely preserving lower bound for G_1 and $s^* \upharpoonright \eta = \tilde{p}$ and $\tilde{p} \Vdash \text{"}s^* \upharpoonright [\eta, \beta) = \dot{s}_1\text{"}$ and $\text{supt}(s^*) \subseteq \eta \cup N_{\beta^*}$. By elementarity, we may assume that G_1 and \dot{s}_1 are elements of N_{α^*} . Because P_β is strictly strongly T -preserving, we may take \dot{s}'_1 such that whenever \tilde{p} is a lower bound for G which is both (M, P_η, T) -completely preserving and $(\langle N_i : \eta^* < i \leq \alpha^* \rangle, P_\eta, T)$ -preserving then $\tilde{p} \Vdash \text{"}\dot{s}'_1 \leq \dot{s}_1\text{"}$ and (\tilde{p}, \dot{s}_1) is $(N_{\alpha^*}, P_\beta, T)$ -preserving and $\tilde{p} \Vdash \text{"}\text{supt}(\dot{s}'_1) \subseteq N_{\alpha^*}[G_{P_\eta}]\text{"}$. Necessarily we have that (\tilde{p}, \dot{s}_1) is (M, P_β, T) -completely preserving. By Lemma 41 we may take \dot{s}_2 and $G' \in \text{Gen}(M, P_\alpha, p)$ such that $G_1 = \{r \upharpoonright \beta : r \in G'\}$ and whenever \tilde{p} is a lower bound for G_1 which is both (M, P_β, T) -completely preserving and $(N_{\alpha^*}, P_\beta, T)$ -preserving then (\tilde{p}, \dot{s}_2) is an (M, P_α, T) -completely preserving lower bound for G' . Let \dot{s} be the P_η -name for the pair (\dot{s}'_1, \dot{s}_2) . Then \dot{s} and G' are as required.

Now we consider the case where α is a limit ordinal. Let $\langle \alpha_n : n \in \omega \rangle$ be an increasing sequence from $\alpha \cap M$ cofinal in $\sup(\alpha \cap M)$ such that $\alpha_0 = \eta$. For every integer $n \geq 0$ let α_n^* be the order-type of $\alpha_n \cap M$. Let $\langle \tau_n : n \in \omega \rangle$ list the set of all P_α -names τ in M such that $\mathbf{1} \Vdash \text{"}\tau \text{ is an ordinal.}"$ Let $\langle \langle x_n, A_n \rangle : n \in \omega \rangle$ list the set of all pairs $\langle x, A \rangle$ such that $x \in T$ and $\text{rk}(x) = \omega_1 \cap M$ and $A \in M$ and A is a P_α -name for a subset of T .

Build $\langle\langle G_n, \dot{s}_n, \dot{s}'_n, p_n \rangle : n \in \omega \rangle$ such that $G_0 = G$ and $p_0 = p$ and each of the following:

- (1) $G_n \in \text{Gen}(M, P_{\alpha_n}, p_n \upharpoonright \alpha_n) \cap N_{\alpha_n^*+1}$
- (2) $p_{n+1} \leq p_n$ and $p_{n+1} \in P_\alpha \cap M$ and $p_{n+1} \upharpoonright \alpha_n \in G_n$ and $p_{n+1} \upharpoonright \alpha_n \Vdash \text{“} p_{n+1} \upharpoonright [\alpha_n, \alpha) \text{ decides the value of } \tau_n \text{, and either } x_n \notin A_n \text{ or there are } y < x_n \text{ and } z \in T \cap M \text{ such that } z \not\leq x_n \text{ and } p_{n+1} \upharpoonright [\alpha_n, \alpha) \Vdash \text{“} \{y, z\} \subseteq A_n \text{.”} \text{”}$
- (3) $\dot{s}_n \in N_{\alpha_{n+1}^*+1}$ and whenever \tilde{p} is a lower bound for G_n and \tilde{p} is both (M, P_{α_n}, T) -completely preserving and $(\langle N_i : \alpha_n^* < i \leq \alpha_{n+1}^* \rangle, P_{\alpha_n}, T)$ -preserving, then there is $\tilde{s} \in P_{\alpha_{n+1}}$ such that $\tilde{s} \upharpoonright \alpha_n = \tilde{p}$ and $\tilde{p} \Vdash \text{“} \tilde{s} \upharpoonright [\alpha_n, \alpha_{n+1}) = \dot{s}_n \text{”}$ and \tilde{s} is an $(M, P_{\alpha_{n+1}}, T)$ -completely preserving lower bound for G_{n+1} .
- (4) $G_n = \{r \upharpoonright \alpha_n : r \in G_{n+1}\}$.
- (5) whenever \tilde{p} is a lower bound for G_n and \tilde{p} is both (M, P_{α_n}, T) -completely preserving and $(\langle N_i : \alpha_n^* < i \leq \alpha^* \rangle, P_{\alpha_n}, T)$ -preserving, then $\tilde{p} \Vdash \text{“} \dot{s}'_n \leq \dot{s}_n \text{”}$ and there is $\tilde{s} \in P_{\alpha_{n+1}}$ such that $\tilde{s} \upharpoonright \alpha_n = \tilde{p}$ and $\tilde{p} \Vdash \text{“} \tilde{s} \upharpoonright [\alpha_n, \alpha_{n+1}) = \dot{s}'_n \text{”}$ and $\text{supt}(\tilde{s}) \subseteq \alpha_n \cup N_{\alpha^*}$ and \tilde{s} is $(\langle N_i : \alpha_{n+1}^* < i \leq \alpha^* \rangle, P_{\alpha_{n+1}}, T)$ -preserving (necessarily, \tilde{s} is $(M, P_{\alpha_{n+1}}, T)$ -completely preserving).

The construction proceeds as follows. Given G_n and p_n , construct p_{n+1} as in (2) as follows. Choose $\dot{q} \in M$ such that $p_n \upharpoonright \alpha_n \Vdash \text{“} \dot{q} \leq p_n \upharpoonright [\alpha_n, \alpha) \text{ and } \dot{q} \text{ decides the value of } \tau_n \text{.”}$ Let $E = \{r \leq p_n \upharpoonright \alpha_n : (\exists s \in P_\alpha)(s \upharpoonright \alpha_n = r \text{ and } r \Vdash \text{“} s \upharpoonright [\alpha_n, \alpha) = \dot{q} \text{”})\}$. Because $E \in M$ we may take $r_1 \in E \cap G_n$. Take $q_1 \in P_\alpha \cap M$ such that $q_1 \upharpoonright \alpha_n = r_1$ and $r_1 \Vdash \text{“} q_1 \upharpoonright [\alpha_n, \alpha) = \dot{q} \text{.”}$ Let $X = \{w \in T : q_1 \not\Vdash \text{“} w \notin A_n \text{”}\}$. We select q_3 as follows. If $x_n \notin X$ let $q_3 = q_1$. Otherwise, take $y < x_n$ such that $y \in X$. We have $r_1 \Vdash \text{“} q_1 \upharpoonright [\alpha_n, \alpha) \not\Vdash \text{“} y \notin A_n \text{”}$ so we may take $\dot{q}_2 \in M$ such that $r_1 \Vdash \text{“} \dot{q}_2 \leq q_1 \upharpoonright [\alpha_n, \alpha) \text{ and } \dot{q}_2 \Vdash \text{“} y \in A_n \text{.”}$ Let $E_1 = \{r \leq r_1 : (\exists s \in P_\alpha)(s \upharpoonright \alpha_n = r \text{ and } r \Vdash \text{“} s \upharpoonright [\alpha_n, \alpha) = \dot{q}_2 \text{”})\}$. Because $E_1 \in M$ we may take $r_2 \in E_1 \cap G_n$. Then take $q_3 \in P_\alpha \cap M$ such that $q_3 \upharpoonright \alpha_n = r_2$ and $r_2 \Vdash \text{“} q_3 \upharpoonright [\alpha_n, \alpha) = \dot{q}_2 \text{.”}$ Let $Y = \{w \in T : q_3 \not\Vdash \text{“} w \notin A_n \text{”}\}$. We build p_{n+1} as follows. If $x_n \notin Y$ then we let $p_{n+1} = q_3$. Otherwise, take $z \in Y \cap M$ such that $z \not\leq x_n$. We have $r_2 \Vdash \text{“} q_3 \upharpoonright [\alpha_n, \alpha) \not\Vdash \text{“} z \notin A_n \text{”}$ so we may take $\dot{q}_4 \in M$ such that $r_2 \Vdash \text{“} \dot{q}_4 \leq q_3 \upharpoonright [\alpha_n, \alpha) \text{ and } \dot{q}_4 \Vdash \text{“} z \in A_n \text{.”}$ Let $E_2 = \{r \leq r_2 : (\exists s \in P_\alpha)(s \upharpoonright \alpha_n = r \text{ and } r \Vdash \text{“} s \upharpoonright [\alpha_n, \alpha) = \dot{q}_4 \text{”})\}$. Because $E_2 \in M$ we may take $r_3 \in E_2 \cap G_n$. Then take $p_{n+1} \in P_\alpha \cap M$ such that $p_{n+1} \upharpoonright \alpha_n = r_3$ and $r_3 \Vdash \text{“} p_{n+1} \upharpoonright [\alpha_n, \alpha) = \dot{q}_4 \text{.”}$

Given p_{n+1} , use the fact that $P_{\alpha_{n+1}}$ is (T, X) -strictly complete to take G_{n+1} and \dot{s}_n as in (1) and (3) and (4). Finally, use Lemma 36 to take \dot{s}'_n as in (5).

Let $G' = \{p \in M : (\exists n \in \omega)(p_n \leq p)\}$, and let \dot{s} be the P_η -name for the

concatenation of $(\dot{s}'_0, \dot{s}'_1, \dots)$, followed by $\mathbf{1}_{\zeta, \alpha}$ where $\zeta = \sup(\alpha \cap M)$.

We show that this choice of G' and \dot{s} works. Given \tilde{p} a lower bound for G which is both (M, P_η, T) -completely preserving and $(\langle N_i : \eta^* < i \leq \alpha^* \rangle, P_\eta, T)$ -preserving, we build $\langle \tilde{p}_n : n \in \omega \rangle$ such that $\tilde{p}_0 = \tilde{p}$ and for every $n \in \omega$ we have $\tilde{p}_{n+1} \restriction \alpha_n = \tilde{p}_n$ and \tilde{p}_{n+1} is a lower bound for G_{n+1} and \tilde{p}_{n+1} is both $(M, P_{\alpha_{n+1}}, T)$ -completely preserving and $(\langle N_i : \alpha_{n+1}^* < i \leq \alpha^* \rangle, P_{\alpha_{n+1}}, T)$ -preserving, and $\tilde{p}_n \Vdash \text{"}\tilde{p}_{n+1} \restriction [\alpha_n, \alpha_{n+1}) = \dot{s}'_n\text{"}$ and $\text{supt}(\tilde{p}_{n+1}) \subseteq \eta \cup N_{\alpha^*}$. This is possible because given \tilde{p}_n , there is a P_{α_n} -name E such that $\tilde{p}_n \Vdash \text{"}E \text{ is a closed subset of } \text{Spec}(\langle N_i : \alpha_n^* < i \leq \alpha^* \rangle, P_{\alpha_n}, T) \text{ of order-type } (\alpha^* + 1) - (\alpha_n^* + 1) \text{ and } (\forall i \in E)(\forall j \in i \cap E)(E \cap N_j \in N_i[G_{P_{\alpha_n}}])\text{"}$. Because $\tilde{p}_n \Vdash \text{"}(\alpha_{n+1}^* + 1) \cap E \text{ has order-type at most } (\alpha_{n+1}^* + 1) - (\eta^* + 1)\text{"}$, we have $\tilde{p}_n \Vdash \text{"}\{i \in E : \alpha_{n+1}^* < i\} \text{ has order-type at least } (\alpha^* + 1) - (\alpha_{n+1}^* + 1)\text{"}$, and hence has order-type exactly equal to $(\alpha^* + 1) - (\alpha_{n+1}^* + 1)$. Hence we may proceed to take \tilde{p}_{n+1} as given above.

Let $r \in P_\alpha$ be such that $\text{supt}(r) \subseteq \sup(\alpha \cap M)$ and $(\forall n \in \omega)(r \restriction \alpha_n = \tilde{p}_n)$. We have $\tilde{p} \Vdash \text{"}r \restriction [\eta, \alpha) = \dot{s}\text{"}$ and r is an (M, P_α, T) -completely preserving lower bound for G' .